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Two Different Rapid Decorrelation in Time Limits for Turbulent Diffusion

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A turbulent diffusion model in which the velocity field is Gaussian and rapidly decorrelating in time (GRDT) has been widely used recently in an endeavor to understand the emergence of anomalous scaling behavior of physical fields in fluid mechanics from the underlying stochastic partial differential equations. The utility of the GRDT model is the fact that correlation functions of the passive scalar field solve closed partial differential equations; the usual moment closure obstacle is averted. We study here the sense in which the GRDT model describes turbulent diffusion by a general, non-Gaussian velocity field with nontrivial temporal structure in the limit in which the correlation time of the velocity field is taken to zero. When the velocity field is rescaled in a particular manner in this rapid decorrelation limit, then a limit theorem of Khas'minskii indeed shows that the passive scalar statistics are described asymptotically by the GRDT Model for a broad class of velocity field models. We provide, however, an explicit example of a "Poisson blob model" velocity field which has two different well-defined rapid decorrelation in time limits. In one, the passive scalar correlation functions converge to those of the GRDT Model, and in the other, they converge to a distinct nontrivial limit in which the correlation functions do not solve closed PDE's. We provide both mathematical and heuristic explanations for the differences between these two limits. The conclusion is that the GRDT Model provides a universal description of the rapid decorrelation in time limit of general non-Gaussian velocity field models only when the velocity field is rescaled in a particular manner during the limit process.

KEY WORDS: Turbulent diffusion; Kraichnan model; Poisson process; convergence of probability measures; Levy–Khinchine theorem; Feynman–Kac formula.

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1. INTRODUCTION

A fundamental obstacle to the analytical understanding of turbulence is the moment closure problem. An attempt to write down an equation of evolution for some statistical moment of the velocity field will, due to the advective nonlinearity, involve statistical moments of higher order.[1, 2] Therefore, one cannot in general obtain a system of closed equations for any finite set of statistical moments of the velocity field. These moments are of interest, however, because they possess anomalous scaling properties. More precisely, moments of velocity differences over some distance scale with the distance variable with some power not predictable by dimensional analysis.[3–5] This raises the question of how these anomalous scaling properties can be understood analytically from the Navier–Stokes equations.

The difficulty of the nonlinearity inherent in the Navier–Stokes equation has led investigators to consider the anomalous scaling problem in the simpler context of a passive scalar field. The evolution of a physical field $T(\mathbf{x}, t)$, such as the concentration density of small particles or small temperature fluctuations, which is passively advected by an incompressible flow can be modeled by the advection-diffusion equation:

$$\frac{\partial T(\mathbf{x}, t)}{\partial t} + \mathbf{v}(\mathbf{x}, t) \cdot \nabla T(\mathbf{x}, t) = \kappa \, \Delta T(\mathbf{x}, t) + f(\mathbf{x}, t),$$
$$T(\mathbf{x}, t = 0) = T_0(\mathbf{x}). \tag{1}$$

where κ is the molecular diffusivity of the passive scalar field and $f(\mathbf{x}, t)$ is an "pumping field" which models an external input of fluctuations in the passive scalar field. The moments of the passive scalar field increments

$$\langle (T(\mathbf{x}+\mathbf{r}, t) - T(\mathbf{x}, t))^N \rangle \tag{2}$$

have also been found to exhibit anomalous scaling for separation distances r in the inertial range.[3, 5] The linearity of the advection-diffusion equation suggests a more promising avenue for an analytical understanding for how a solution to a stochastic PDE should exhibit anomalous scaling. There would of course be no simplification of the analysis if the velocity field $\mathbf{v}(\mathbf{x}, t)$ were to be taken as the solution to the randomly driven Navier–Stokes equations. Instead, one models the $\mathbf{v}(\mathbf{x}, t)$ as a random field with prescribed statistics which mimic in at least some ways the observed spatio-temporal structure of fully developed, homogenous, isotropic turbulence. The pumping field $f(\mathbf{x}, t)$ is also typically taken to be described by a statistically isotropic random field fluctuating on large scales relative to the inertial range.

The moment closure problem arises again, however, because an attempt to deduce a PDE for the Nth order passive scalar (PS) correlation function

$$P_N(\{\mathbf{x}^{(j)}\}_{j=1}^{N}, t) \equiv \left\langle \prod_{j=1}^{N} T(\mathbf{x}^{(j)}, t) \right\rangle$$

from the advection-diffusion equation (1) will involve higher order correlation functions of the form

$$\left\langle \mathbf{v}(\mathbf{x}^{(j')}, t) \prod_{j=1}^{N} T(\mathbf{x}^{(j)}, t) \right\rangle,$$

and again there is no general way to close the hierarchy of equations for the correlation functions. This moment closure problem manifests the hidden nonlinearity of the advection-diffusion equation (1): while it is indeed linear in $T(\mathbf{x}, t)$ for every realization of $\mathbf{v}(\mathbf{x}, t)$, the statistics of the solution $T(\mathbf{x}, t)$ depend nonlinearly on the statistics of the random coefficient $\mathbf{v}(\mathbf{x}, t)$.

There does however exist a special random velocity field model for which the turbulent diffusion closure problem can be averted. Suppose the velocity field $\mathbf{v}(\mathbf{x}, t)$ is a Gaussian random field which is mean zero, statistically homogenous, and delta-correlated in time:

$$\langle \mathbf{v}(\mathbf{x}, t) \rangle = 0,$$

$$\langle \mathbf{v}(\mathbf{x}, t) \otimes \mathbf{v}(\mathbf{x}+\mathbf{r}, t+\tau) \rangle = \mathscr{R}(\mathbf{r})\, \delta(\tau).$$

Suppose further that the random pumping is also a Gaussian random field which is mean zero, statistically homogenous, and delta-correlated in time:

$$\langle f(\mathbf{x}, t) \rangle = 0,$$

$$\langle f(\mathbf{x}, t) \otimes f(\mathbf{x}+\mathbf{r}, t+\tau) \rangle = \Phi(\mathbf{r})\, \delta(\tau).$$

We call this the *Gaussian Rapid Decorrelation in Time* (GRDT) Model for the advection-diffusion of a passive scalar field. It is often called the Kraichnan model after one of its original proposers.[6] (Kazantsev[7] independently suggested such a model for a magnetohydrodynamic turbulent flow.) With this GRDT Model, one can derive a closed hierarchy of PDE's for the correlation functions of the passive scalar field to arbitrary order:

$$\frac{\partial P_N(\{\mathbf{x}^{(i)}\}_{i=1}^N, t)}{\partial t}$$

$$= \mathscr{M}_N P_N(\{\mathbf{x}^{(i)}\}_{i=1}^N, t) + \frac{1}{2} \sum_{j \neq j'} \Phi(\mathbf{x}^{(j)} - \mathbf{x}^{(j')}) \, P_{N-2}(\{\mathbf{x}^{(i)}\}_{i \neq j, j'}, t). \quad (3a)$$

with the differential operators:

$$\mathscr{M}_N \equiv \kappa \sum_{j=1}^N \Delta_j + \frac{1}{2} \sum_{j, j'=1}^N \nabla_j \cdot (\mathscr{R}(\mathbf{x}^{(j)} - \mathbf{x}^{(j')}) \cdot \nabla_{j'}). \quad (3b)$$

It is to be understood that $P_{-1} \equiv 0$ and $P_0 \equiv 1$. ∇_j and Δ_j denote differentiation with respect to the coordinates of the jth particle. These equations are recursively solvable, in the sense that they may in principle be solved one by one without ever having to contend with a simultaneous system of multiple PDE's. The mean statistics $P_1(\mathbf{x}, t)$ may first be solved since they are decoupled. Then the equation for the second order statistics $P_2(\mathbf{x}, t)$ may be solve since it is coupled only to $P_1(\mathbf{x}, t)$, which has just been obtained. And one can continue to recursively solve for P_N using the solutions for $P_1 \cdots P_{N-1}$. This simple coupling of the PS correlation functions is completely different from that which arises in standard turbulence theory, where the equation for P_N requires knowledge of a higher order statistical quantity rather than a lower order one.

The possibility of writing down closed equations for the correlation functions in the GRDT Model was pointed out in refs. 6, 8, and 9. Majda[10] used these equations to describe the higher-order statistics of freely decaying passive scalar fluctuations within the inertial range of a turbulent GRDT shear flow model. Kraichnan[11] then advanced arguments that the passive scalar increments (2) should exhibit anomalous scaling within the GRDT Model, igniting vigorous activity by several research groups to elucidate these anomalous scaling properties directly from the exact equations (3b) for the PS correlation functions in the GRDT Model (see refs. 12–16 and other references in ref. 17). Similar Gaussian velocity field models with rapid decorrelation in time have been explored to study properties of the tails of the single-point distribution for the passive scalar field.[18–23] Other properties of passive scalar fields in the GRDT Model are examined in refs. 23–25.

There exist a variety of formal derivations of the equations (3b) and special cases thereof.[23, 24, 26–29] A fundamental difficulty in achieving a rigorous derivation is the need to make sense of the advection-diffusion PDE (1) with a random coefficient $\mathbf{v}(\mathbf{x}, t)$ which only exists as a generalized random field, due to its delta-correlated nature. Perhaps the most satisfactory way to make rigorous sense of the GRDT Model without introducing approximating sequences is to interpret the velocity field as a Brownian flow[30] and to represent the solution to the advection-diffusion equation in terms of the statistics of the trajectories of tracers moving through a Brownian flow and undergoing an additional independent Brownian motion due to molecular diffusion.[31, 32]

In the original work,[6] Kraichnan interpreted the GRDT Model as describing a limit of a velocity field with short but finite correlation time. Our aim in this paper is to add some rigorous clarification to this point of view. Some work along these lines has been accomplished by Majda[33] and Molchanov *et al.,*[8, 9] who obtained the equations for the passive scalar correlation functions in the GRDT Model (with no pumping) as a limiting description for a certain class of velocity field "renewal" models as the correlation time was taken to zero while the amplitude of the velocity was rescaled to infinity. We will show in Section 2 that in fact the GRDT Model equations do generally arise as a rapid decorrelation in time limit of a broad class of random non-Gaussian velocity field models for $\mathbf{v}(\mathbf{x}, t)$ with nontrivial temporal correlations, *provided that this rapid decorrelation in time limit is performed according to the following rescaling*:

$$\begin{aligned} \mathbf{v}^{(\varepsilon)}(\mathbf{x}, t) &= \varepsilon^{-1/2}\mathbf{v}(\mathbf{x}, t/\varepsilon), \\ f^{(\varepsilon)}(\mathbf{x}, t) &= \varepsilon^{-1/2} f(\mathbf{x}, t/\varepsilon). \end{aligned} \tag{4}$$

with $\varepsilon \searrow 0$. The example of refs. 8, 9, and 33 does fall in this class. One may be tempted from this fact to conclude that the GRDT Model universally describes the advection of a passive scalar field by a velocity field with very short correlation time. That is, one might suppose that the specification that the GRDT velocity field is Gaussian is gratuitous, since the equations of the GRDT model also describe the short correlation time limit of a large class of non-Gaussian models.

Our aim in this paper is to scrutinize this notion of universality in the rapid decorrelation limit in the context of a *Poisson Blob Shear Flow* Model,[34] which we will define in Section 3. We shall show that the manner in which the rapid decorrelation limit of a given velocity field is taken can strongly influence the limiting behavior of the passive scalar statistics. We will explicitly compute two distinct limits of the PS correlation functions arising from the Poisson blob Model which each correspond to advection by a random velocity field with rapid decorrelations in time. Moreover, the second order correlation functions of the velocity field coincide in the two limit processes. Each yields a nontrivial limit for the PS correlation functions, one of which corresponds to the GRDT Model, while the other is manifestly different. Thus, when discussing the effects of a rapidly decorrelating velocity field on a passive scalar, one must be careful to specify how the short correlation time limit of the velocity field is to be interpreted. Thus, the rapid decorrelation limit of the passive scalar statistics is *not* universal with respect to different ways of taking the zero correlation time limit.

The veteran probabilist can understand this outcome via the Levy–Khinchine theorem (see Section 5.5). A velocity field in any zero correlation time limit should generate a flow with independent increments. In particular, tracers advected by the flow should be described by a process with independent increments. In a homogenous, stationary flow, a *single* tracer will moreover move according to a process with stationary, independent increments. Such processes are completely characterized by the Levy–Khinchine theorem. The key fact is that such a process is a combination of a mean drift, a Brownian motion, and a generalized Poisson type motion. In rapid decorrelation limits which converge to the GRDT model, the tracer trajectories converge to Brownian motions. But in the rapid decorrelation limit of the Poisson blob model which produces a distinct limiting behavior, the tracer trajectories have a Poissonian component. The derivation of the GRDT model equations in refs. 31 and 32 specifically require that the tracers diffuse according to Brownian motion processes, and does not carry over when the tracers have Poissonian behavior. Indeed, one can show by our simple example that the passive scalar correlation functions in the alternative zero correlation time limit are not governed by a PDE, but rather by a pseudo-differential evolution equation (Section 5).

The sense in which the Gaussian Rapid Decorrelation in Time Model does universally describe the statistical behavior of passive scalar fields advected by non-Gaussian velocity field models with nontrivial temporal structure *under the particular limit process (4) with correlation time tending to zero* is presented in Section 2. In Section 3, we define the Poisson Blob Shear Flow model and define two rescalings leading to different rapid decorrelation in time limits. A discussion of the different physics of the two limits follows in Section 4. The mathematical statement of the limiting behavior of the PS correlation functions is presented and contrasted in Section 5. The computations and justifications may be found in Sections 6, 7, and 8. A review of some basic properties of the Poisson point process which we will need is presented in Appendix A. Some auxiliary lemmas are stated in Appendix B.

To avoid distractions from our main point of interest, we consider only smooth velocity field models (with two continuous derivatives), such as velocity field models corresponding to positive viscosity.[13, 17, 34, 35] Fractal velocity field models with no dissipation scale cutoff, as are often used in anomalous scaling studies[13, 26, 27, 36–41] are only Hölder continuous and do not satisfy the smoothness conditions assumed in the present work. However, we expect that the smoothness conditions we impose are not essential, and that fractal velocity field models would also exhibit, in close analogy, multiple distinct rapid decorrelation in time limits.

Let us also stress that the behavior which we derive for the tracer trajectories in the short correlation time limit of the velocity field is *not* equivalent to fixing the temporal structure of the velocity field and then looking at the tracer trajectories at long times. The reason is that the rapid decorrelation in limits which we consider (such as (4)) involve no rescaling of the spatial scales. By contrast, the evolution of tracer trajectories in a given velocity field will at long times generally involve large spatial scales.[34, 35, 42, 43]

2. UNIVERSALITY OF GRDT MODEL UNDER DIFFUSIVE RESCALING LIMIT

We address here the positive sense in which the RDT model equations describe not only the statistical behavior of the passive scalar field in a truly delta-correlated velocity and pumping field environment, but also the behavior in models with finite correlation times, in the limit that the correlation time is sent to zero. Suppose we are given a model velocity $\mathbf{v}(\mathbf{x}, t)$ and pumping field $f(\mathbf{x}, t)$ which are statistically homogenous in space, stationary in time, and have mean zero, but which may be non-Gaussian and decorrelate in time at a finite rate. One can show that under the *particular* rescaling:

$$\begin{aligned} \mathbf{v}^{(\varepsilon)}(\mathbf{x}, t) &\equiv \varepsilon^{-1/2}\mathbf{v}(\mathbf{x}, t/\varepsilon), \\ f^{(\varepsilon)}(\mathbf{x}, t) &\equiv \varepsilon^{-1/2} f(\mathbf{x}, t/\varepsilon) \end{aligned} \tag{5}$$

the evolution of the passive scalar correlation functions associated to these rescaled fields converge as $\varepsilon \searrow 0$ to functions obeying the GRDT model PDE's, provided certain technical conditions are met (see below). The relationship between the spatial correlation structures $\mathscr{R}(\mathbf{r})$ and $\Phi(\mathbf{r})$ of the limiting GRDT Model velocity and pumping fields are related to the second order correlation functions of these fields in the original model:

$$\begin{aligned} \tilde{\mathscr{R}}(\mathbf{r}, \tau) &= \langle \mathbf{v}^{(\varepsilon)}(\mathbf{x}, t) \otimes \mathbf{v}^{(\varepsilon)}(\mathbf{x}+\mathbf{r}, t+\tau)\rangle, \\ \tilde{\Phi}(\mathbf{r}, \tau) &= \langle f(\mathbf{x}, t)\, f(\mathbf{x}+\mathbf{r}, t+\tau)\rangle, \end{aligned} \tag{6a}$$

in the following way:

$$\mathscr{R}(\mathbf{r}) = \int_{-\infty}^{\infty} \tilde{\mathscr{R}}(\mathbf{r}, \tau)\, d\tau,$$

$$\Phi(\mathbf{r}) = \int_{-\infty}^{\infty} \tilde{\Phi}(\mathbf{r}, \tau)\, d\tau. \tag{6b}$$

The rescaled velocity and pumping fields vary on a time scale ε, so the $\varepsilon \searrow 0$ limit is equivalent to rescaling the correlation time of these fields to zero. The amplitude rescaling is necessary for these fields to have a non-trivial effect on the passive scalar field in the $\varepsilon \to 0$ limit; see Section 4. The particular rescaling displayed in (5) will be called a *diffusive* rescaling because the amplitude of the velocity field is rescaled according to the general link between space ℓ and time τ scales in diffusion processes:

$$\tau \to \lambda\tau,$$

$$\ell \to \lambda^{1/2}\ell,$$

$$v \to \lambda^{-1/2}v,$$

The latter transformation is suggested by the dimensions of the velocity field as length divided by time. Note that the diffusive rescaling does *not* alter the spatial argument of the velocity field statistics, so the spatial correlation length (if it exists) remains untouched. We refer to the limiting behavior of the passive scalar field in a given model under the $\varepsilon \searrow 0$ limit with the velocity and pumping field as rescaled in (5) as the *diffusive rapid decorrelation in time (DRDT)* Limit.

2.1. Rigorous DRDT Convergence Criterion

We formulate now a theorem providing conditions on the velocity field which rigorously guarantees that the PS correlation functions converge to solutions of the GRDT Model in the DRDT Limit in the absence of pumping $f(\mathbf{x}, t) = 0$. There is a natural formal extension to include pumping, but extra technicalities enter which we do not wish to dwell on here.[31]

One natural condition required for convergence to the GRDT Model under the rescaling (5) is that the velocity field obey a certain "mixing condition" which effectively guarantees that the velocity field in the unrescaled model loses memory at some sufficiently rapid (but finite) rate. To describe one frequently used measure of mixing,[44] we introduce the probability measure P and corresponding σ-field $\mathscr{F}$ of measurable sets on the underyling abstract probability space Ω. We further define the filtrations $\{\mathscr{F}_{s,s'}\}$ for $s \leqslant s'$ as the σ-subfield of $\mathscr{F}$ generated by the restriction of the random velocity field $\mathbf{v}(\mathbf{x}, t)$ to the interval $s \leqslant t \leqslant s'$.[45] Colloquially speaking, the filtration $\mathscr{F}_{s,s'}$ comprises those events which depend on the velocity field only through its behavior over the time interval $s \leqslant t \leqslant s'$. We can now define the *uniform mixing rate*

$$\phi(t) = \sup_{s \geqslant 0} \sup_{A \in \mathscr{F}_{s+t,\infty},\, B \in \mathscr{F}_{0,s}} |P(A \mid B) - P(A)|,$$

where $P(A \mid B)$ denotes the conditional expectation of event A given event B. The rate of decay of $\phi(t)$ describes how quickly the velocity field loses memory. $\phi(t)$ vanishes over times $t \geqslant \tau$ in renewal models, such as those considered by Majda[33] and Molchanov *et al.*,[8, 9] where the velocity field completely refreshes after a certain time interval τ.

Theorem 1. The PS correlation functions corresponding to a mean zero, incompressible velocity field with general statistics (and zero pumping $f(\mathbf{x}, t) \equiv 0$) converge under the DRDT scaling (5) to the solutions of the GRDT Model equations with coefficients determined by Eq. (6b), provided that:

1. For every $M > 0$,

$$\left\langle \sup_{|\mathbf{x}| \leqslant M} \left[|\mathbf{v}(\mathbf{x}, t)|^2 + \|\nabla \mathbf{v}(\mathbf{x}, t)\|^2 + \|\nabla\nabla \mathbf{v}(\mathbf{x}, t)\|^2 \right] \right\rangle < \infty,$$

2. the velocity field has the following uniform mixing property:

$$\int_0^\infty (\phi(s))^{1/2}\, ds < \infty,$$

3. the initial PS correlation functions $P_{N,0}(\{\mathbf{x}^{(j)}\})$ are bounded and continuous.

This theorem follows by applying a technical improvement (ref. 46, Theorem 4.2) of Khas'minskii's classical limit theorem[47] for ODE's with random coefficients with fast time dependence to the system of stochastic differential equations:

$$d\mathbf{X}^{(\varepsilon),(j)}(s) = \varepsilon^{-1/2} \mathbf{v}(\mathbf{X}^{(\varepsilon),(j)}(s), s/\varepsilon)\, ds + \sqrt{2\kappa}\, d\mathbf{W}^{(j)}(s)$$

describing the joint trajectories $\{\mathbf{X}^{(\varepsilon),(j)}(t)\}_{j=1}^N$ of N particles advected by the rescaled velocity field, where $\{\mathbf{W}^{(j)}(t)\}_{j=1}^N$ are N independent Brownian motion (Wiener) processes. This limit theorem implies, under suitable mixing and smoothness conditions that the limiting behavior of the N particles as $\varepsilon \searrow 0$ is governed by a coupled Brownian motion process. The statistical laws for the joint motion a system of N particles can be mapped to an evolution law for the Nth order PS correlation function $P_N(\{\mathbf{x}^{(j)}\}, t)$ through the theory of Itô diffusion processes.[31, 48] From this, one infers the GRDT Model equations for the PS correlation functions (3b). We remark that the limit theorem of ref. 46 actually needs to be applied to the compensated trajectories $\mathbf{X}^{(\varepsilon),(j)}(s) - \sqrt{2\kappa}\, \mathbf{W}^{(j)}(s)$, which solve a modified random ordinary differential equation without a white noise term $d\mathbf{W}^{(j)}(s)$; the limiting behavior of these compensated trajectories will then of course imply a corresponding limit for the original trajectories $\mathbf{X}^{(\varepsilon),(j)}(s)$. A variation of Theorem 3 can be formulated to apply to velocity field models with weaker mixing but stronger smoothness assumptions by adaptation of Theorem 4.6 in ref. 46.

Another way of deriving propositions concerning the limiting behavior of the PS correlation functions under the rescaling (5) is through direct estimates on the advection-diffusion PDE in a "parametrix" approach; see refs. 8 and 49. Another analytical PDE approach is presented in ref. 50 for the case in which the original unscaled velocity field $\mathbf{v}(\mathbf{x}, t)$ is Markovian in time and Gaussian.

2.2. Heuristic Perspective

Note that the diffusive rapid decorrelation in time (DRDT) limit of the PS correlation functions stated above involves only the mean and second order statistics of the original random velocity field. The higher order statistics are asymptotically irrelevant. The reason for this result may be explained in terms of a "central limit theorem in the environment." Let us explain this phrase. The key notion behind the central limit theorem is that the probability distribution of a sum of a large number of independent, identically distributed random variables has an asymptotically universal (Gaussian) form which depends only on the mean and variance of the random variables, i.e., the first and second order statistics. Consider now the environment felt by a tracer particle moving through a fluid in which the velocity and pumping is rapidly decorrelating in time at some small value of ε in (5). Over any short, finite time interval, the tracer will feel a large number of independent pushes by the velocity field because the correlation time of the velocity field is very small. These pushes will be roughly identically distributed over sufficiently short time intervals (independent of ε) so that the tracer hasn't moved too much. The cumulative influence of these pushes is thus like a sum of a large $O(\varepsilon^{-1})$ number of independent random variables. One might thus expect a central limit theorem to hold here, so that the influence of the random velocity field on a tracer particle over a short time interval is captured entirely by the mean and second order statistics of the random velocity field.

This argument is readily generalized to the joint motion of a finite collection of particles in that over small time intervals (independent of ε), the displacements of the tracer particles should be *approximately* described by a jointly Gaussian distribution. The reason why this statement is not exact when considering the joint motion of a collection of particles is that their motions are coupled due to their response to the spatially correlated velocity flow field $\mathbf{v}(\mathbf{x}, t)$, and this coupling depends on their relative separation which of course changes as they move. Thus, over time scales such that the particles move a distance comparable to their separation, the central limit theorem argument does not apply since the independent random *relative* pushes on two particles from the velocity blobs will not be identically distributed over the whole time interval. But by considering sufficiently small time intervals (independent of ε), one does deduce from the central limit theorem argument that the response of the tracer particles to the velocity field depends only on the mean and second order statistics.

None of the above arguments suggest that the passive scalar field itself is Gaussian in the DRDT limit, and such a supposition is false. Note also that we are not discussing a homogenization result, ref. 17, Section 2, because the spatial variable $\mathbf{x}$ is not rescaled.

3. TWO DIFFERENT RAPID DECORRELATION LIMIT PROCESSES FOR POISSON BLOB SHEAR FLOW MODEL

In the remainder of the paper, we will show that the universality result for the GRDT Model discussed in Section 2 does *not* imply that the GRDT Model describes all rapid decorrelation in time limits of passive scalar field advection-diffusion models. The convergence to the GRDT Model is guaranteed through use of Theorem 3 only if the rapid decorrelation limits of the velocity and pumping fields are taken through the diffusive rescalings (5). It will suffice for our purposes to consider the case of a freely decaying passive scalar field with no pumping $f(\mathbf{x}, t) = 0$ in what follows.

3.1. Poisson Blob Shear Flow Model

We now introduce a *Poisson Blob Shear Flow* model, developed by Avellaneda and Majda,[34] to illustrate explicitly the possibility of a distinct rapid decorrelation in time limit for which the passive scalar field statistics are not asymptotically described by the GRDT Model. The velocity field in this model is taken to be a random, two-dimensional shear flow:

$$\mathbf{v}(x, y, t) = \begin{bmatrix} 0 \\ v(x, t) \end{bmatrix}. \tag{7}$$

The statistics of the shearing component are described as follows. Define ϕ to be some spatio-temporal "blob" structure function with zero integral:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, t)\, dx\, dt = 0. \tag{8}$$

To minimize technical considerations, we further suppose that ϕ is smooth (C^2 is amply sufficient) with compact support. The shear flow is then built out of a superposition of these spatio-temporal blob functions, with centers distributed according to a Poisson point process $(\xi_\lambda^{(n)}, \tau_\lambda^{(n)})$ in $\mathbb{R} \times \mathbb{R}$ with intensity λ:

$$v(x, t) = \sum_n \phi(x - \xi_\lambda^{(n)}, t - \tau_\lambda^{(n)}) \tag{9}$$

Roughly speaking, a Poisson point process of intensity λ lays down points randomly so that the expected number of points in any domain of area A is λA, and the number of points appearing in any two disjoint domains are statistically independent. See Appendix A.1 for a formal definition. Further discussion may be found in ref. 51.

3.2. Two Different Rescalings of Poisson Blob Model

We next explicitly construct two different one-parameter rescalings of a given Poisson blob model for the velocity field. Each will have a natural parametric limit process associated to the correlation time becoming arbitrarily small.

3.2.1. Diffusive Rescaling

The family $v_{\mathrm{D}}^{(\varepsilon)}(x, t)$ of Poisson blob models with *diffusive rescaling* is defined:

$$v_{\mathrm{D}}^{(\varepsilon)}(x, t) = \sum_n \phi_{\mathrm{D}}^{(\varepsilon)}(x - \xi_{\varepsilon^{-1}}^{(n)}, t - \tau_{\varepsilon^{-1}}^{(n)}) \tag{10}$$

where the intensity of the Poisson point process $(\xi_{\varepsilon^{-1}}^{(n)}, \tau_{\varepsilon^{-1}}^{(n)})$ is taken as ε^{-1}, and the rescaled blob functions are defined:

$$\phi_{\mathrm{D}}^{(\varepsilon)}(x, t) = \varepsilon^{-1/2} \phi(x, t/\varepsilon), \tag{11}$$

where ϕ is some fixed prototype blob function satisfying the technical conditions stated above. The reason we call this a diffusive rescaling is because it is statistically equivalent to rescaling (5) discussed in Section 2. To see this, write:

$$v_{\mathrm{D}}^{(\varepsilon)}(x,t)=\varepsilon^{-1/2}\phi(x-\xi_{\varepsilon^{-1}}^{(n)},\,t/\varepsilon-\tau_{\varepsilon^{-1}}^{(n)}/\varepsilon).$$

Here $(\xi_{\varepsilon^{-1}}^{(n)},\tau_{\varepsilon^{-1}}^{(n)})$ denotes a Poisson point process of intensity ε^{-1}. By stretching the distribution of points along the τ direction: $(\xi_{\varepsilon^{-1}}^{(n)},\varepsilon^{-1}\tau_{\varepsilon^{-1}}^{(n)})$, it is easily checked from the characterizing features of a Poisson point process (Appendix A) that a Poisson point process of unit intensity $(\tilde{\xi}^{(n)},\tilde{\tau}^{(n)})$ results. Hence, the statistics of $v_{\mathrm{D}}^{(\varepsilon)}(x,t)$ may be equivalently described:

$$v_{\mathrm{D}}^{(\varepsilon)}(x,t)=\varepsilon^{-1/2}\phi(x-\tilde{\xi}^{(n)},\,t/\varepsilon-\tilde{\tau}^{(n)})=\varepsilon^{-1/2}v(x,t/\varepsilon), \tag{12}$$

as we desired to show. From Theorem 3 in Section 2, we can deduce that the passive scalar statistics resulting from the Poisson blob velocity field rescaled as in (10) will be well described by the GRDT model as $\varepsilon\searrow 0$. (The required mixing condition of the velocity field is easily satisfied due to the compact support of the Poisson blobs.) To facilitate comparison with the other rapid decorrelation in time rescaling which we will consider, we provide a proof of convergence of the Poisson blob Model to the GRDT model under the diffusive rescaling (10) through explicit computation in Section 7.

3.2.2. Fixed Intensity Rescaling

The family of Poisson blob models with *fixed intensity* rescaling,

$$v_{\mathrm{FI}}^{(\varepsilon)}(x,t)=\sum_n \phi_{\mathrm{FI}}^{(\varepsilon)}(x-\xi^{(n)},\,t-\tau^{(n)}), \tag{13}$$

is defined with the intensity of the Poisson point process $(\xi^{(n)},\tau^{(n)})$ fixed at unity and the amplitude of the blob function rescaled from the prototype ϕ in a fashion different from Eq. (11):

$$\phi_{\mathrm{FI}}^{(\varepsilon)}(x,t)=\varepsilon^{-1}\phi(x,t/\varepsilon). \tag{14}$$

The label "D" will refer to various random fields associated with the diffusive rescaling and the label "FI" will refer to the fixed intensity rescaling. At $\varepsilon=1$, both models coincide with the unscaled Poisson blob velocity field:

$$v(x,t)=\sum_n \phi(x-\xi^{(n)},\,t-\tau^{(n)})$$

where the Poisson point process $(\xi^{(n)}, \tau^{(n)})$ has unit intensity. We call the limiting behavior of the passive scalar statistics in the $\varepsilon \to 0$ limit under diffusive rescaling the *diffusive rapid decorrelation in time (DRDT)* limit, and under fixed intensity rescaling the *fixed intensity rapid decorrelation in time (FIRDT)* limit.

3.3. Equivalence of Mean and Second Order Statistics of Velocity Field in Two Limits

We now show that, for any ε, the mean and second order correlation function of the Poisson blob model velocity field coincide under the two rescalings, and therefore have identical $\varepsilon \searrow 0$ rapid decorrelation in time limits. The mean of all rescaled velocity fields vanish identically because of Corollary 11 in Appendix A and the zero integral condition (8) on the blob functions ϕ. Let

$$\tilde{R}(x, t) = \langle v(x', t')\, v(x+x', t+t') \rangle$$

denote the correlation function of the random velocity field, which is easily checked to be statistically homogenous in space and stationary in time.[52] Then using Corollary 12 in Appendix A, we find:

$$\begin{aligned}
\tilde{R}(x, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x', t')\, \phi(x+x', t+t')\, dx'\, dt', \\
\tilde{R}_{\mathrm{D}}^{(\varepsilon)}(x, t) &\equiv \langle v_{\mathrm{D}}^{(\varepsilon)}(x', t')\, v_{\mathrm{D}}^{(\varepsilon)}(x+x', t+t') \rangle \\
&= \varepsilon^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{\mathrm{D}}^{(\varepsilon)}(x', t')\, \phi_{\mathrm{D}}^{(\varepsilon)}(x+x', t+t')\, dx'\, dt' \\
&= \varepsilon^{-1} \tilde{R}(x, t/\varepsilon), \\
\tilde{R}_{\mathrm{FI}}^{(\varepsilon)}(x, t) &\equiv \langle v_{\mathrm{FI}}^{(\varepsilon)}(x', t')\, v_{\mathrm{FI}}^{(\varepsilon)}(x+x', t+t') \rangle \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{\mathrm{FI}}^{(\varepsilon)}(x', t')\, \phi_{\mathrm{FI}}^{(\varepsilon)}(x+x', t+t')\, dx'\, dt' \\
&= \varepsilon^{-1} \tilde{R}(x, t/\varepsilon). \qquad (15)
\end{aligned}$$

Thus, the mean and second order statistics of the Poisson blob velocity field coincide under both rescalings, and the second order correlation function moreover converges to a delta-correlated form in the sense of generalized functions as $\varepsilon \searrow 0$:

$$\lim_{\varepsilon \searrow 0} \tilde{R}_{\mathrm{FI}}^{(\varepsilon)}(x, t) = \lim_{\varepsilon \searrow 0} \tilde{R}_{\mathrm{D}}^{(\varepsilon)}(x, t) = R(x)\, \delta(t),$$

$$R(x) \equiv \int_{-\infty}^{\infty} \tilde{R}(x, t)\, dt \tag{16}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x', t')\, \phi(x + x', t + t')\, dx'\, dt'\, dt$$

We note for future reference that $\tilde{R}(x, t)$ and $R(x)$ are clearly smooth and of compact support since the blob functions ϕ have these properties.

Although the first and second order statistics of Poisson blob velocity fields under both rescalings have been found to agree, the higher order statistics of the velocity fields must differ because the rescalings are different. Indeed, in Section 5, we exhibit the exact statistics of a freely decaying passive scalar field advected by a Poisson blob velocity field in both the DRDT and FIRDT limits. We show that the limiting PS correlation functions obey different evolution laws. In the DRDT limit, they will will obey the diffusion PDE's of the GRDT model, but in the FIRDT limit, they will not be described by the solution to an (evident) PDE.

4. PHYSICAL EXPLANATION OF TWO RAPID DECORRELATION LIMITS

In the hope of clarifying the mathematical presentation in the succeeding sections, we offer here a heuristic picture of the advection of a tracer in the small ε limit under both diffusive and fixed intensity rescalings of the Poisson blob model.

We aim to establish concretely three points in this section.

1. A central limit in the environment should be expected in the DRDT limit (Section 4.1). We have already discussed this in a general context in Section 2, but we want to give a physical picture here for the Poisson blob model to contrast with the FIRDT limit.

2. In the FIRDT limit, the tracer motion is intermittent and does not have a central limit character in the $\varepsilon \searrow 0$ limit (Section 4.2).

3. Finally, we give in Section 4.3 a heuristic explanation for how the second order correlation function for $v(x, t)$ can have the same $\varepsilon \to 0$ limit under diffusive and fixed intensity rescaling when the nature of the advection is so radically different.

4.1. Diffusive Rapid Decorrelation in Time (DRDT) Limit

Under diffusive rescaling of a Poisson blob model,

$$v_{\mathrm{D}}^{(\varepsilon)}(x,t)=\sum_{n}\phi_{\mathrm{D}}^{(\varepsilon)}(x-\xi_{\varepsilon^{-1}}^{(n)},t-\tau_{\varepsilon^{-1}}^{(n)}),$$

$$\phi_{\mathrm{D}}^{(\varepsilon)}(x,t)=\varepsilon^{-1/2}\phi(x,t/\varepsilon)$$

the $\varepsilon\searrow 0$ limit thins the blob function in the temporal direction and simultaneously increases the intensity of the Poisson point process $\tau^{(n)}$ by a factor of ε^{-1}. So as a tracer moves along during some fixed interval of time, it will feel the effects of $O(\varepsilon^{-1})$ different blobs. While the tracer remains within the confines of a given blob, it zooms through according to the local value of the velocity, and exits the blob at some pretty much randomized location. As the centers of the blobs are independently distributed, we can suppose that, in the small ε limit, we can approximately treat the advection of a tracer a sum of individual pushes from each blob it encounters.

Note that the tracer will be within any given blob for a typical time of order ε, which is the temporal width of the rescaled blob $\phi_{\mathrm{D}}^{(\varepsilon)}$. (The time to escape a blob by exiting its spatial domain is much larger (of order unity) because the tracer motion through molecular diffusion in the x direction does not speed up as $\varepsilon\to 0$). Within the blob, the tracer is advected by a velocity field of magnitude scaling as $\varepsilon^{-1/2}$, and its motion is *correlated* while it remains in a given blob. Hence, the random distance which a tracer will move due to an encounter with a given blob will be of order $\varepsilon^{1/2}$. More precisely, this random distance will have mean zero and variance proportional to ε. Summing up the effects of ε^{-1} pushes from independent blobs over an order unity time interval, we have an order unity total mean-squared displacement. The DRDT limit thus leads to finite advection, and the net displacement over order unity time intervals is roughly given by a sum of a large number of independent kicks. This is exactly the kind of situation in which a central limit theorem should apply, and the motion of a tracer should be well described by a Brownian motion. This is why we call this limit process “diffusive.”

4.2. Fixed Intensity Rapid Decorrelation in Time (FIRDT) Limit

Under the fixed intensity rescaling, the intensity of Poisson blobs is held fixed while the blob functions are thinned in the temporal direction. The fraction of space-time covered by Poisson blobs in a typical realization will thus eventually scale proportionally to ε under fixed intensity rescaling.

The velocity field will therefore be *quiescent* over most of the space-time domain when ε is small. Consider then a tracer particle advected by a Poisson blob velocity field rescaled under fixed intensity at small ε. Most of the time, the turbulence will be inactive and the tracer diffuses purely through molecular means. But over an order unity duration of time, it will encounter an order unity number of blobs which rapidly shove it. The time a tracer will spend in any given blob is proportional to ε, and the velocity magnitude of a blob under fixed intensity rescaling is proportional to ε^{-1}, so the tracer will feel an order unity displacement from each blob encounter. Since there is an order unity number of such kicks, we expect a nontrivial limit for passive scalar advection in the $\varepsilon \to 0$ limit under fixed intensity rescaling.

There is no reason to expect here the manifestation of a Central Limit Theorem as there was for the DRDT limit. In an order unity time interval, there are only order unity independent advection events (encounters with a blob) in the FIRDT limit, even as $\varepsilon \to 0$. The velocity field is very intermittent (admittedly unrealistically so), and its effect on the passive scalar will be rather peculiar. It would be wrong, as we shall see, to model it as an effective diffusion.

A schematic comparison of a fixed intensity and diffusive rescaling of the Poisson blob velocity field is shown in Fig. 1.

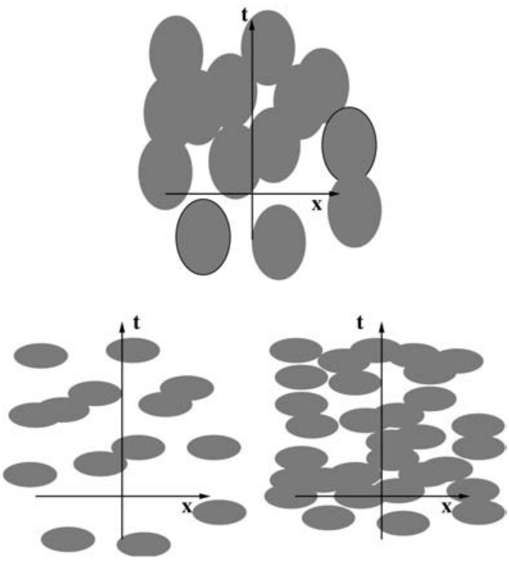

Fig. 1. Top: Support of unscaled Poisson blobs. Left hand side: Support of Poisson blobs distributed according to a fixed intensity rescaling. Right hand side: Support of Poisson blobs distributed according to a diffusive (D) rescaling.

4.3. Coincidence of Second Order Correlation Functions in Two Limits

It may seem surprising that the Poisson blob velocity field obeys the same second order statistics for the two rather different rapid decorrelation in time limiting procedures (see (15)):

$$\begin{aligned}\langle v_{\mathrm{D}}^{(\varepsilon)}(x',t')\, v_{\mathrm{D}}^{(\varepsilon)}(x+x',t+t')\rangle &= \langle v_{\mathrm{FI}}^{(\varepsilon)}(x',t')\, v_{\mathrm{FI}}^{(\varepsilon)}(x+x',t+t')\rangle \\ &= \varepsilon^{-1}\tilde{R}(x,t/\varepsilon).\end{aligned}$$

We showed this mathematically in Section 3, but wish to give a physical explanation to provide further clarification of the rescalings.

The fact that the velocity field is becoming delta-correlated in time in the $\varepsilon \searrow 0$ limit under both diffusive and fixed intensity rescaling is clear because the blobs are being thinned. As the spatial structure of the velocity field is fixed in either limit process, it will suffice for pedagogical purposes to explain why the variance of the velocity field scales as ε^{-1} for both limits. Under diffusive rescaling, it is easily understood why the variance of the velocity field scales as ε^{-1}: the velocity field amplitude is scaled as $\varepsilon^{-1/2}$ and its mean square should scale as ε^{-1}. Under fixed intensity rescaling, on the other hand, the amplitude of the velocity field is scaled as ε^{-1}, but the velocity field is only active a fraction of the time proportional to ε. Thus the variance of the Poisson blob velocity field should rather be estimated as ε (on the order of the probability that the velocity field is actually active at a given space-time location) multiplied by ε^{-2} (the mean-square amplitude of the velocity field when it is active), and this yields ε^{-1}. Hence, the single-point variance of the velocity field scales, under both diffusive and fixed intensity rescaling, in inverse proportion to the effective correlation time ε of the velocity field.

5. PASSIVE SCALAR STATISTICS IN RAPID DECORRELATION LIMITS

We now present the explicit mathematical difference between the evolution of the passive scalar statistics in the DRDT and FIRDT limits of the Poisson blob velocity field. We set up in Section 5.1 by introducing the tracer particle trajectories associated with the advection-diffusion equation in the Poisson blob model, and present the general explicit link between the Nth order passive scalar correlation function and the joint statistics of the motion of N tracer particles. We report in turn the passive scalar statistics for the DRDT and FIRDT limits in Sections 5.2 and 5.3, deferring the derivations for later sections. We then discuss in Section 5.4 the nature of the limiting statistics for each case, and contrast them. In Section 5.5, we refer to the Levy–Khinchine theorem which sheds some light on the reason for the different behavior of the passive scalar field in the two rapid decorrelation in time limits.

The Poisson blob velocity field supplies a very clean illustration of the fact that the statistical behavior of the passive scalar statistics in the limit of short temporal correlations in the velocity field can depend very strongly on how the limit is interpreted.

5.1. Tracer Particle Trajectories

It will be useful, both in the derivation and presentation of the results, to introduce the *N-particle transition density*

$$p_N(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t), \tag{17}$$

which is defined as the joint probability density of the spatial distribution of the locations of N tracers $\{(X^{(j)}(t'), Y^{(j)}(t')\}_{j=1}^{N}$ which were released at positions

$$(X^{(j)}(t), Y^{(j)}(t)) = (x^{(j)}, t^{(j)}) \tag{18a}$$

at time $t' = t$. These trajectories are determined from the stochastic differential equations:

$$\begin{aligned} dX^{(j)}(s) &= \sqrt{2\kappa}\, dW_x^{(j)}(s), \\ dY^{(j)}(s) &= v(X^{(j)}(s), s)\, ds + \sqrt{2\kappa}\, dW_y^{(j)}(s), \end{aligned} \tag{18b}$$

The $\{W_x^{(j)}(s), W_y^{(j)}(s)\}_{j=1}^{N}$ are a collection of independent Wiener processes (or Brownian motions), ref. 48, pp. 7–10 representing the effects of molecular diffusion. The Wiener process $W(s)$ is characterized as a continuous random process with $W(0) = 0$, and all increments $W(s) - W(s')$ governed by a Gaussian distribution with zero mean and variance

$$\langle (W(s) - W(s'))^2 \rangle = |s - s'|.$$

Note that the statistics of the tracer trajectories are coupled through the common random coefficient v. The N-particle transition density is mathematically defined so that:

$$\begin{aligned}&\mathrm{Prob}\{(X^{(j)}(t'), Y^{(j)}(t')) \in A_j, j=1,..., N \mid \\ &\qquad X^{(j)}(t)=x^{(j)}, Y^{(j)}(t)=y^{(j)}, j=1,..., N\} \\ &\quad = \int_{A_1\times\cdots\times A_N} p_N(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}_{j=1}^N\, d\{y'^{(j)}\}_{j=1}^N\end{aligned}$$

for any Borel sets $A_j \in \mathbb{R}^2$. (Braces without indices, such as $\{x^{(j)}, y^{(j)}\}$ are implicitly defined to refer to the collection in which j ranges from 1 to N.) It can be shown, by using the statistical properties of the Poisson point process, the smoothness of the blob functions, and the first Borel–Cantelli lemma, ref. 53, Section 4, that almost every realization of the random Poisson blob shear velocity field will satisfy suitable smoothness and growth conditions so that the theory of parabolic PDE's[54, 55] and diffusion processes[56] guarantees that the N-particle transition density (17) exists as a continous function for $t' \neq t$.[31]

The N-particle transition density acts as a Green's function for the Nth order PS correlation function[31]

$$\begin{aligned}P_N(\{x^{(j)}, y^{(j)}\}, t) = \int_{\mathbb{R}^{2N}} & p_N(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t) \\ & \times P_{N,0}(\{x'^{(j)}, y'^{(j)}\})\, d\{x'^{(j)}\}_{j=1}^N\, d\{y'^{(j)}\}_{j=1}^N, \qquad (19)\end{aligned}$$

where

$$P_{N,0}(\{x'^{(j)}, y'^{(j)}\}) \equiv P_N(\{x'^{(j)}, y'^{(j)}\}, t=0)$$

is the initial data for the PS correlation function, which we assume to be smooth and bounded. Note that the N-particle transition density appearing in (19) involves the statistics of tracer trajectories moving backwards in time. This is a consequence of the backwards Kolmogorov equation (ref. 48, p. 105) which relates the solution of the parabolic advection-diffusion equation (1) (with $f=0$) to an average over statistics of tracer trajectories moving backward in time. It is also possible to write down a "forwards" transition density representation for P_N,[31] but this would be less amenable to our method of analysis. To describe the evolution of the PS correlation functions P_N, it clearly suffices to describe the evolution of the N-particle transition density p_N. We will therefore focus our attention on the N-particle transition density since it is more fundamental. The connection between the statistics of the Nth order passive scalar correlation function P_N and the joint statistics of N tracer trajectories was exploited in various applications in refs. 10, 15, 16, 31, and 57.

We call the Fourier transform of p_N with respect to the target variables $\{x^{(j)}, y'^{(j)}\}$:

$$\hat{p}_N(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$$

$$\equiv \int_{\mathbb{R}^{2N}} d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N}$$

$$\times\, p_N(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)\, e^{2\pi i(\sum_{j=1}^{N} \eta^{(j)} x'^{(j)} + k^{(j)} y'^{(j)})}$$

the N*-particle characteristic function*; it is indeed by definition equal to

$$\left\langle e^{2\pi i \sum_{j=1}^{N} (\eta^{(j)} X^{(j)}(t') + k^{(j)} Y^{(j)}(t'))} \right\rangle,$$

where the trajectories have initial conditions (18a). The averaging here is over both the velocity field statistics and the Wiener processes.

5.2. Diffusive Rapid Decorrelation in Time Limit

Recall the definition of the diffusively rescaled Poisson blob velocity field ((10) and (11)):

$$v_{\mathrm{D}}^{(\varepsilon)}(x, t) = \varepsilon^{-1/2} \sum_{n} \phi(x - \xi_{\varepsilon^{-1}}^{(n)}, (t - \tau_{\varepsilon^{-1}}^{(n)})/\varepsilon), \tag{20}$$

with $(\xi_{\varepsilon^{-1}}^{(n)}, \tau_{\varepsilon^{-1}}^{(n)})$ a Poisson point process of intensity ε^{-1} on $\mathbb{R} \times \mathbb{R}$. We indicated in (12) that this definition is equivalent to the general diffusive rescaling (5). Let $T_{\mathrm{D}}^{(\varepsilon)}(x, y, t)$ denote the (random) solution to the advection-diffusion equation governed by the velocity field (20),

$$\begin{gathered} \frac{\partial T_{\mathrm{D}}^{(\varepsilon)}(x, y, t)}{\partial t} + v_{\mathrm{D}}^{(\varepsilon)}(x, t)\, \frac{\partial T_{\mathrm{D}}^{(\varepsilon)}(x, y, t)}{\partial y} = \kappa\, \Delta T_{\mathrm{D}}^{(\varepsilon)}(x, y, t), \\ T_{\mathrm{D}}^{(\varepsilon)}(x, y, t = 0) = T_0(x, y). \end{gathered} \tag{21}$$

and let $P_{N,\mathrm{D}}^{(\varepsilon)}$ be the associated PS correlation functions, $p_{N,\mathrm{D}}^{(\varepsilon)}$ be the N-particle transition densities of the associated particle trajectories, and $\hat{p}_{N,\mathrm{D}}^{(\varepsilon)}$ be the corresponding N-particle characteristic functions.

The limiting behavior of these statistical quantities are described in the following proposition:

Proposition 2 (DRDT Limit). Under the diffusive rescaling (20):

1. Each N-particle characteristic function converges pointwise

$$\lim_{\varepsilon \searrow 0} \hat{p}_{N,\mathrm{D}}^{(\varepsilon)}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) = \hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t),$$

where the limiting N-particle characteristic function is the unique classical solution to the PDE:

$$\frac{\partial \hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)}{\partial t}$$

$$= \kappa \sum_{j=1}^{N} \Delta_j \hat{\bar{p}}_{N,\mathrm{D}} + U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{x^{(j)}\})\, \hat{\bar{p}}_{N,\mathrm{D}}, \tag{22}$$

$$\hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t = t') = e^{2\pi i \sum_{j=1}^{N} (\eta^{(j)} x^{(j)} + k^{(j)} y^{(j)})}.$$

for $t' < t$, where the "potential" $U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{x^{(j)}\})$ is given by:

$$U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{x^{(j)}\}) = -2\pi^2 \sum_{m=1}^{N} \sum_{n=1}^{N} k^{(m)} k^{(n)} R(x^{(m)} - x^{(n)}). \tag{23}$$

The function $R(x)$ is defined as the integral over a constant x slice of the second order spatio-temporal correlation function of the original Poisson blob model:

$$R(x) = \int_{-\infty}^{\infty} \tilde{R}(x, t)\, dt.$$

2. The N-particle transition density converges weakly as a probability measure (ref. 53, Section 25):

$$p_{N,\mathrm{D}}^{(\varepsilon)}(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N}$$

$$\Rightarrow \bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N},$$

where the limit $\bar{p}_{N,\mathrm{D}}$ is an N-particle transition density corresponding to the GRDT Model, with evolution equation for $t' < t$:

$$\frac{\partial \bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)}{\partial t}$$

$$= \kappa \sum_{j=1}^{N} \Delta_j \bar{p}_{N,\mathrm{D}} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} R(x^{(i)} - x^{(j)}) \frac{\partial^2 \bar{p}_{N,\mathrm{D}}}{\partial y^{(i)}\, \partial y^{(j)}}, \tag{24a}$$

$$\bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t' = t) \tag{24b}$$

$$= \prod_{j=1}^{N} \delta(x'^{(j)} - x^{(j)})\, \delta(y'^{(j)} - y^{(j)}).$$

Moreover, $\hat{\bar{p}}_{N,\mathrm{D}}$ is the N-particle characteristic function corresponding to the N-particle transition density $\bar{p}_{N,\mathrm{D}}$.

3. Each correlation function $P^{(\varepsilon)}_{N,\mathrm{D}}(\{x^{(j)}, y^{(j)}\}, t)$ converges uniformly over compact sets to a continuous function:

$$\lim_{\varepsilon \searrow 0} P^{(\varepsilon)}_{N,\mathrm{D}}(\{x^{(j)}, y^{(j)}\}, t) = \bar{P}_{N,\mathrm{D}}(\{x^{(j)}, y^{(j)}\}, t) \tag{25}$$

which may be expressed in terms of the limiting N-particle transition density in the usual manner:

$$\bar{P}_{N,\mathrm{D}}(\{x^{(j)}, y^{(j)}\}, t) = \int_{\mathbb{R}^{2N}} \bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t) \\ \times P_{N,0}(\{x'^{(j)}, y'^{(j)}\})\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N}.$$

The limiting PS correlation functions $\bar{P}_{N,\mathrm{D}}$ obey the same diffusion PDE's (24a) as $\bar{p}_{N,\mathrm{D}}$.

To be clear, the differential operator Δ_j has the following meaning:

$$\Delta_j \equiv \frac{\partial^2}{\partial (x^{(j)})^2} + \frac{\partial^2}{\partial (y^{(j)})^2}.$$

The proof of the proposition will be supplied in Section 7. We separately stated the limiting behavior of the N-particle characteristic function to facilitate comparison with the FIRDT limit, to which we turn next.

5.3. Fixed Intensity Rapid Decorrelation Limit

We now state the evolution of the passive scalar statistics in the rapid decorrelation limit under fixed intensity rescaling ((13) and (14)):

$$v^{(\varepsilon)}_{\mathrm{FI}}(x, t) = \varepsilon^{-1} \sum_n \phi\left(x - \xi^{(n)}, \frac{t - \tau^{(n)}}{\varepsilon}\right) \tag{26}$$

where $(\xi^{(n)}, \tau^{(n)})$ is a Poisson point process of unit intensity on $\mathbb{R} \times \mathbb{R}$. Let $T^{(\varepsilon)}_{\mathrm{FI}}(x, y, t)$ denote the (random) solution to the advection-diffusion equation governed by this Poisson blob velocity field rescaled with fixed intensity:

$$\frac{\partial T^{(\varepsilon)}_{\mathrm{FI}}(x, y, t)}{\partial t} + v^{(\varepsilon)}_{\mathrm{FI}}(x, t) \frac{\partial T^{(\varepsilon)}_{\mathrm{FI}}(x, y, t)}{\partial y} = \kappa\, \Delta T^{(\varepsilon)}_{\mathrm{FI}}(x, y, t), \\ T^{(\varepsilon)}_{\mathrm{FI}}(x, y, t) = T_0(x, y, t), \tag{27}$$

and let $P^{(\varepsilon)}_{N,\,\mathrm{FI}}$ be the associated PS correlation functions, $p^{(\varepsilon)}_{N,\,\mathrm{FI}}$ be the N-particle transition densities of the associated particle trajectories, and $\hat{p}^{(\varepsilon)}_{N,\,\mathrm{FI}}$ be the corresponding N-particle characteristic functions.

Then we have the following proposition about the FIRDT limiting behavior of the passive scalar statistics:

Proposition 3 (FIRDT Limit). Under the fixed intensity rescaling (26),

1. Each N-particle characteristic functions converges pointwise

$$\lim_{\varepsilon \searrow 0} \hat{p}^{(\varepsilon)}_{N,\,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$$
$$= \hat{\bar{p}}_{N,\,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t),$$

where the limiting N-particle characteristic function is the unique classical solution to the PDE:

$$\frac{\partial \hat{\bar{p}}_{N,\,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)}{\partial t}$$
$$= \kappa \sum_{j=1}^{N} \Delta_j \hat{\bar{p}}_{N,\,\mathrm{FI}} + U_{N,\,\mathrm{FI}}(\{k^{(j)}\}, \{x^{(j)}\})\, \hat{\bar{p}}_{N,\,\mathrm{FI}}, \tag{28}$$
$$\hat{\bar{p}}_{N,\,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t = t') = e^{2\pi i \sum_{j=1}^{N} (\eta^{(j)} x^{(j)} + k^{(j)} y^{(j)})}.$$

for $t' < t$, where the "potential" $U_{N,\,\mathrm{FI}}(\{k^{(j)}\}, \{x^{(j)}\})$ is given by:

$$U_{N,\,\mathrm{FI}}(\{k^{(j)}\}, \{x^{(j)}\}) = \int_{-\infty}^{\infty} (e^{-2\pi i \sum_{l=1}^{N} k_l \bar{\phi}(x_l - \xi)} - 1)\, d\xi \tag{29}$$

The function $\bar{\phi}(x)$ is defined as the integral over constant x slices of the Poisson blob function:

$$\bar{\phi}(x) = \int_{-\infty}^{\infty} dt\, \phi(x, t). \tag{30}$$

2. The N-particle transition density converges weakly as a probability measure:

$$p^{(\varepsilon)}_{N,\,\mathrm{FI}}(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N}$$
$$\Rightarrow \bar{p}_{N,\,\mathrm{FI}}(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N},$$

where the limit $\bar{p}_{N,\mathrm{FI}}$ is an N-particle transition density. Moreover, $\hat{\bar{p}}_{N,\mathrm{FI}}$ is the N-particle characteristic function corresponding to the N-particle transition density $\bar{p}_{N,\mathrm{FI}}$.

3. Each correlation function $P^{(\varepsilon)}_{N,\mathrm{FI}}(\{x^{(j)}, y^{(j)}\}, t)$ converges uniformly over compact sets to a continuous function:

$$\lim_{\varepsilon \searrow 0} P^{(\varepsilon)}_{N,\mathrm{FI}}(\{x^{(j)}, y^{(j)}\}, t) = \bar{P}_{N,\mathrm{FI}}(\{x^{(j)}, y^{(j)}\}, t) \tag{31}$$

which may be expressed in terms of the limiting N-particle transition density in the usual manner:

$$\bar{P}_{N,\mathrm{FI}}(\{x^{(j)}, y^{(j)}\}, t) = \int_{\mathbb{R}^{2N}} \bar{p}_{N,\mathrm{FI}}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t) \times P_{N,0}(\{x'^{(j)}, y'^{(j)}\})\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N}.$$

The proof will be given in Section 8.

5.4. Contrast of FIRDT and DRDT Limits

The import of the propositions just reported is that the FIRDT limit of the passive scalar statistics differ vastly from the DRDT limit in the Poisson blob model, even though the second order correlation functions of the velocity field are identical in the two rapid decorrelation in time limits.

The limiting N-particle transition density in the DRDT limit, $\bar{p}_{N,\mathrm{D}}$, satisfies a diffusion PDE, as does the limiting N-point PS correlation function $\bar{P}_{N,\mathrm{D}}$. These are just the GRDT model equations (see Section 3.2) for the special case in which the velocity field is a shear flow and there is no pumping. Indeed, when the velocity field is diffusively rescaled to a rapid decorrelation in time limit, the tracer trajectories converge in distribution to coupled Brownian motions. The diffusive rescaling thus leads to a diffusive limiting behavior of the passive scalar which is well described by the GRDT model equations.

On the other hand, the FIRDT limiting passive scalar statistics cannot be expressed as the solution of a PDE. To see this, we consider the PDE's (22) and (28) for the limiting N-particle characteristic function. Both have the form of a Schrödinger equation in imaginary time, with potentials

$$U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{x^{(j)}\}) = -2\pi^2 \sum_{m=1}^{N} \sum_{n=1}^{N} k^{(m)} k^{(n)} R(x^{(m)} - x^{(n)}),$$

$$U_{N,\mathrm{FI}}(\{k^{(j)}\}, \{x^{(j)}\}) = \int_{-\infty}^{\infty} (e^{-2\pi i \sum_{l=1}^{N} k_l \bar{\phi}(x_l - \xi)} - 1)\, d\xi.$$

The important difference in these potentials is that the DRDT potential $U_{N,\mathrm{DR}}$ is quadratic in $\{k^{(j)}\}$ while $U_{N,\mathrm{FI}}$ is some transcendental function of $\{k^{(j)}\}$. Indeed, it is easy to see that $\hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$ and $\hat{\bar{p}}_{N,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$ each remain proportional to $e^{2\pi i \sum_{j=1}^{N} k^{(j)} y^{(j)}}$ multiplied by some function independent of $\{y^{(j)}\}$. Hence, multiplication of $\hat{\bar{p}}_{N,\mathrm{D}}$ by $2\pi i k^{(j)}$ is equivalent to differentiation with respect to $y^{(j)}$. The fact that $U_{N,\mathrm{DR}}$ is quadratic in $\{k^{(j)}\}$ means that it is equivalent to a second order differential operator, which yields the second order diffusion PDE (24c) for $\bar{p}_{N,\mathrm{D}}$. On the other hand, under the transformation $2\pi i k^{(j)} \to \frac{\partial}{\partial y^{(j)}}$, $U_{N,\mathrm{FI}}$ does not become a simple differential operator since it is not a simple polynomial in $\{k^{(j)}\}$. It becomes instead some sort of pseudo-differential operator.[58] One could in this way endeavor to write down a pseudo-differential evolution equation for the FIRDT limiting transition density $\bar{p}_{N,\mathrm{FI}}$ and the limiting PS correlation functions $\bar{P}_{N,\mathrm{FI}}$, but we do not pursue this.

Since the first and second order statistics for the Poisson blob velocity field are identical in the FIRDT and DRDT limits, the difference between the limiting passive scalar statistics in the two scenarios must be due to the higher order statistics of the velocity field. Note moreover that the DRDT limiting statistics depends on the blob function ϕ only through the second order statistics of the velocity field, as manifested in $R(x)$ (see Section 3.3). The higher order statistics of the velocity field are thus irrelevant for the DRDT limit, so they must be relevant for the FIRDT limit.

A clear way to understand this outcome is that a "central limit theorem in the environment" is active in the DRDT limit, but not in the FIRDT limit. We discussed in Section 4.2 that in the FIRDT limit, a tracer particle will only encounter an order unity number of blobs per time interval, and thus no central limit theorem behavior could be expected. In fact, the advection process was seen to be highly intermittent.

In the DRDT limit, on the other hand, a sample tracer will encounter many small kicks in an order unity time interval, leading to a diffusive behavior governed by the central limit theorem. As far as the passive scalar is concerned, the velocity field is effectively Gaussian in the DRDT limit. This can be seen explicitly by noting that, via the Feynman–Kac formula (stated in Appendix B), the DRDT limiting N-particle characteristic function has the form, for $t' < t$:

$$\begin{aligned}
&\hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t) \\
&\quad = e^{2\pi i (\sum_{j=1}^{N} k^{(j)} y^{(j)}) - 4\pi^2 \kappa \sum_{j=1}^{N} (k^{(j)})^2 (t-t')} \\
&\qquad \times \left\langle e^{2\pi i \sum_{j=1}^{N} \eta^{(j)} X^{(j)}(s)} e^{-2\pi^2 \sum_{m,n=1}^{N} k^{(m)} k^{(n)} \int_{t'}^{t} R(X^{(m)}(s) - X^{(n)}(s))\, ds} \right\rangle_W
\end{aligned}$$

where:

$$X^{(j)}(t') = x^{(j)} + \sqrt{2\kappa}\, W_x^{(j)}(t-t')$$

the $\{W_x^{(j)}(s)\}$ are independent Wiener processes, and $\langle \cdot \rangle$ denotes an average over the statistics of the Wiener processes. On the other hand, the N-particle characteristic function associated to a general mean zero, homogenous, stationary, Gaussian random shear velocity field with correlation function:

$$\tilde{R}_G(x,t) = \langle v(x',t')\, v(x'+x, t'+t)\rangle$$

is given, for $t' < t$, by[35]:

$$\begin{aligned}
&\hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t) \\
&\quad = e^{2\pi i (\sum_{j=1}^N k^{(j)} y^{(j)}) - 4\pi^2 \kappa \sum_{j=1}^N (k^{(j)})^2 (t-t')} \\
&\qquad \times \left\langle e^{2\pi i \sum_{j=1}^N \eta^{(j)} X^{(j)}(t')} e^{-2\pi^2 \sum_{m,n=1}^N k^{(m)} k^{(n)} \int_{t'}^t \int_{t'}^t \tilde{R}_G(X^{(m)}(s) - X^{(n)}(s'),\, s-s')\, ds\, ds'} \right\rangle_W .
\end{aligned}$$

Thus, we manifestly see that the DRDT limit of the N-particle characteristic function is exactly that which would be obtained by advection by a Gaussian random shear flow which is delta-correlated in time:

$$\tilde{R}_G(x,t) = R(x)\, \delta(t).$$

5.5. Levy–Khinchine Theorem Perspective

The reason for the existence of distinct passive scalar behavior in various rapid decorrelation limits of the velocity field can be understood by consideration of the Levy–Khinchine theorem, see ref. 59, Section 3.2. Loosely, this theorem says that a stationary random process of bounded variation with no memory (i.e., with independent increments) is a combination of a mean drift, a Brownian motion, and a generalized version of a Poisson process. The important point for our purposes is that Brownian motion is not the only random process which has no memory.

It can be checked, using standard compactness arguments (as can be found in ref. 60) that the statistical paths of N tracers in the Poisson blob velocity field converge in distribution, under both diffusive and fixed intensity rescaling, to certain random processes with N-particle transition density given by $\bar{p}_{N,\mathrm{D}}$ and $\bar{p}_{N,\mathrm{FI}}$, respectively. (Convergence occurs in the space of continuous paths in the DRDT limit but only in Skorohod space in general in the FIRDT limit; see ref. 60). We know that $\bar{p}_{N,\mathrm{D}}$ satisfies the diffusion equation (24c) corresponding to the GRDT model, so in the DRDT limit, the tracers move according to a coupled Brownian motion.[31] On the other hand, the transition density $\bar{p}_{N,\mathrm{FI}}$ of the FIRDT limiting tracer motion does not obey the GRDT model rules. Consider the one-particle transition density $\bar{p}_{1,\mathrm{FI}}$ and the corresponding one-particle characteristic function for which we have the exact formula, for $t' < t$:

$$\hat{\bar{p}}_{1,\mathrm{FI}}(\eta, k; t' | x, y, t)$$
$$= \exp\left(-4\pi^2\kappa(\eta^2 + k^2)(t - t') + (t - t') \int_{-\infty}^{\infty} (e^{2\pi i k \bar{\phi}(\xi)} - 1)\, d\xi \right), \qquad (32)$$

it is easily shown that the motion of a single tracer in the FIRDT limit has independent increments, as it should. Upon comparison of (32) with the discussion of the Levy–Khinchine theorem in ref. 59, Section 3.2, it is seen that the FIRDT limiting motion of the tracer in the shearing direction (y) is governed by a superposition of Brownian motion from molecular diffusion and a generalized Poisson process from the Poisson blob velocity field. Using our assumptions on the compact support of the Poisson blob function ϕ, it can further be shown that this generalized Poisson process for the single tracer motion is simply a piecewise constant random process with jump times distributed according to a Poisson point process of finite intensity, with each jump independently chosen from a fixed distribution. (To put (32) in the canonical Levy–Khinchine form for the characteristic function of a stationary random process with independent increments, simply change variables in the integral from ξ to $z = \phi(\xi)$). Note, by the way, that the *joint* motion of N tracers is not described in either the DRDT or FIRDT limit by an independent increment process because their relative separation rate depends on their current relative separation.

We can now satisfactorily understand why the Poisson blob shear velocity field has two distinct limiting advection behaviors. Under a diffusive rescaling as in Section 2, a central limit theorem in the environment applies and the Poisson blob velocity field acts very much like a Gaussian delta-correlated velocity field. On the other hand, in the FIRDT limit, the tracer always feels the Poisson point process underlying the blob distribution, even as the effective correlation time becomes zero. Consequently, the passive scalar statistics do not converge to those of the GRDT model.

We next turn to the derivations of the propositions stated at the beginning of this section. To prepare, we first derive a general formula for the passive scalar statistics in unscaled Poisson blob Model in Section 6 and then consider the DRDT and FIRDT limits in Sections 7 and 8.

6. PASSIVE SCALAR STATISTICS IN POISSON BLOB MODEL

The fundamental statistical object we will analyze is the N-particle characteristic function, which can most concisely be expressed by:

$$\hat{p}_N(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) = \langle e^{2\pi i \sum_{j=1}^{N} (\eta^{(j)} X^{(j)}(t') + k^{(j)} Y^{(j)}(t'))} \rangle, \tag{33}$$

where the tracer particle trajectories in a shear flow 7 are given by the stochastic differential equations (18b), which may be explicitly integrated:

$$X^{(j)}(t') = x^{(j)} + \sqrt{2\kappa}\, W_x^{(j)}(t-t'), \tag{34a}$$

$$Y^{(j)}(t') = y^{(j)} - \int_{t'}^{t} v(X^{(j)}(s), s)\, ds + \sqrt{2\kappa}\, W_y^{(j)}(t-t'), \tag{34b}$$

where $\{W_x^{(j)}(s), W_y^{(j)}(s)\}_{j=1}^{N}$ is a collection of independent Wiener processes. For the Poisson blob model statistics for the shear flow, the N-particle characteristic function may be expressed in an almost explicit form, as first demonstrated in ref. 34:

Proposition 4. For the Poisson blob shear flow velocity field defined as in Section 3.1, the N-particle characteristic function is given by the following formula:

$$\begin{aligned}
&\hat{p}_N(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) \\
&\quad = e^{2\pi i (\sum_{j=1}^{N} k^{(j)} y^{(j)}) - 4\pi^2 \kappa \sum_{j=1}^{N} (k^{(j)})^2 (t-t')} \Bigg\langle e^{2\pi i \sum_{j=1}^{N} \eta^{(j)} X^{(j)}(t')} \\
&\qquad \times \exp\left[\lambda \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \left(e^{-2\pi i \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \phi(X^{(m)}(s) - \xi,\, s-\tau)} - 1\right)\right] \Bigg\rangle_W
\end{aligned} \tag{35}$$

where

$$X^{(j)}(s) = x^{(j)} + \sqrt{2\kappa}\, W^{(j)}(t-s), \tag{36}$$

and $\langle \cdot \rangle_W$ denotes a statistical average over $\{W^{(j)}(s)\}_{j=1}^{N}$.

6.1. Proof of Formula for N-Particle Characteristic Function

Substituting the expressions for $Y^{(j)}(t')$ from (18a) into the definition (33) of the N-particle characteristic function, we obtain:

$$\hat{p}_N(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$$
$$= \langle e^{2\pi i(\sum_{j=1}^N \eta^{(j)} X^{(j)}(t')) + 2\pi i(\sum_{j=1}^N k^{(j)} y^{(j)})}$$
$$\times e^{-2\pi i \sum_{j=1}^N k^{(j)} \int_{t'}^t v(X^{(j)}(s), s)\, ds} e^{-2\sqrt{2\kappa}\,\pi i \sum_{j=1}^N k^{(j)} W_y^{(j)}(t-t')} \rangle. \tag{37}$$

This expression is to be averaged over the statistics of the velocity field as well as the statistics of the Wiener processes $\{W_x^{(j)}(t)\}_{j=1}^N$ and $\{W_y^{(j)}(t)\}_{j=1}^N$. The averaging over the Wiener processes $W_y^{(j)}(t)$ may be performed explicitly, noting that these processes only appear in the last factor:

$$\langle e^{-2\sqrt{2\kappa}\,\pi i \sum_{j=1}^N k^{(j)} W_y^{(j)}(t-t')} \rangle = e^{-4\pi^2 \kappa \sum_{j=1}^N (k^{(j)})^2 |t-t'|}.$$

Here we used the defining properties of the Wiener process (stated after Eq. (18b)), and the formula for the characteristic function of a Gaussian random variable Z with mean μ and variance σ^2, ref. 53, p. 348:

$$\langle e^{2\pi i \lambda Z} \rangle = e^{2\pi i \lambda \mu} e^{-2\pi^2 \lambda^2 \sigma^2}.$$

Next, the averaging over the Poisson statistics underlying the velocity field, which only appears in the penultimate factor in Eq. (37), can be performed through an application of the exact formula for the characteristic functional of a Poisson point process (Proposition 10 in Appendix A.2):

$$\langle e^{-2\pi i \sum_{j=1}^N k^{(j)} \int_{t'}^t v(X^{(j)}(s), s)\, ds} \rangle_v$$
$$= \langle e^{-2\pi i \sum_{j=1}^N k^{(j)} \int_{t'}^t \sum_n \phi(X^{(j)}(s) - \xi^{(n)}, s - \tau^{(n)})\, ds} \rangle_v$$
$$= \exp\left[\lambda \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \left(e^{-2\pi i \sum_{j=1}^N k^{(m)} \int_{t'}^t ds\, \phi(X^{(j)}(s) - \xi, s - \tau)} - 1\right) \right].$$

The realization of the random processes $\{X^{(j)}(s)\}_{j=1}^N$ were held fixed during this averaging over only the velocity statistics. The formula stated in the proposition now follows.

7. DERIVATION OF PASSIVE SCALAR STATISTICS IN DRDT LIMT

Here we derive the limiting behavior of the passive scalar statistics advected by a Poisson blob velocity field in the DRDT limit, as reported in Proposition 2. This proposition may be viewed as a corollary of Theorem 3 stated in Section 2 concerning the convergence of passive scalar statistics to those of the GRDT Model under a rapid decorrelation in time limit with diffusive rescaling of general "mixing" velocity field models. But to provide an explicit computation against which to compare the FIRDT limit, we will give a direct, self-contained proof utilizing the exact formula for the PS correlation functions of a Poisson blob shear velocity field which were developed in Section 6. Our proof generalizes the one given for the mean passive scalar density ($N=1$) in ref. 34.

Recall that under diffusive rescaling, the Poisson blob intensity is ε^{-1} and the rescaled blob function is (11):

$$\phi_{\mathrm{D}}^{(\varepsilon)}(x, t) = \varepsilon^{-1/2}\phi(x, t/\varepsilon).$$

The N-particle characteristic function $\hat{p}_{N,\mathrm{D}}^{(\varepsilon)}$ associated to the diffusively rescaled Poisson blob velocity field then has the following explicit formula (for $t' < t$), according to Proposition 4:

$$\begin{aligned}
&\hat{p}_{N,\mathrm{D}}^{(\varepsilon)}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) \\
&\quad = e^{2\pi i(\sum_{j=1}^{N} k^{(j)} y^{(j)}) - 4\pi^2 \kappa \sum_{j=1}^{N} (k^{(j)})^2 (t-t')} \Big\langle e^{2\pi i \sum_{j=1}^{N} \eta^{(j)} X^{(j)}(t')} \\
&\qquad \times \exp\Big[\varepsilon^{-1}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \big(e^{-2\pi i \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \varepsilon^{-1/2}\phi(X^{(m)}(s)-\xi, (s-\tau)/\varepsilon)} - 1\big)\Big]\Big\rangle_W .
\end{aligned} \tag{38}$$

where:

$$X^{(j)}(s) = x^{(j)} + \sqrt{2\kappa}\, W^{(j)}(t-s). \tag{39}$$

The main part of the derivation, presented in Section 7.1, is showing that $\hat{p}_{N,\mathrm{D}}^{(\varepsilon)}$ converges pointwise to the limiting N-particle characteristic function:

$$\begin{aligned}
&\hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) \\
&\quad \equiv e^{2\pi i(\sum_{j=1}^{N} k^{(j)} y^{(j)}) - 4\pi^2 \kappa \sum_{j=1}^{N} (k^{(j)})^2 (t-t')} \\
&\qquad \times \Big\langle e^{2\pi i \sum_{j=1}^{N} \eta^{(j)} X^{(j)}(t')} \exp\Big[\int_{t'}^{t} U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{X^{(j)}(s)\})\, ds\Big]\Big\rangle_W .
\end{aligned} \tag{40}$$

Here the "potential" is defined:

$$U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{x^{(j)}\}) = -2\pi^2 \sum_{m=1}^{N} \sum_{n=1}^{N} k^{(m)} k^{(n)} R(x^{(m)} - x^{(n)})$$

with $R(x)$ given by (16).

The PDE's for $\hat{\bar{p}}_{N,\mathrm{D}}$ and $\bar{p}_{N,\mathrm{D}}$ are then shown to follow by the Feynman–Kac formula in Section 7.2. Finally, we mop up some technical details concerning convergence in Section 7.3.

7.1. Convergence of *N*-Particle Characteristic Function in DRDT Limit

We aim here to show that:

Claim 5. For $t' < t$,

$$\lim_{\varepsilon \searrow 0} \hat{p}^{(\varepsilon)}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) = \hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$$

pointwise, with $\hat{p}^{(\varepsilon)}_{N,\mathrm{D}}$ given by (38) and $\hat{\bar{p}}_{N,\mathrm{D}}$ given by (40).

To begin with, the $\varepsilon \searrow 0$ limit may certainly be brought inside the expectation over Wiener processes by the bounded convergence theorem.

As we are anticipating a central limit theorem type result, we are naturally led to Taylor expand complex exponentials through quadratic order, as one does in the Fourier analysis proof of the ordinary central limit theorem. Using Taylor's theorem with remainder, we have the following calculus inequality:

$$|e^{2\pi i x} - (1 + 2\pi i x - 2\pi^2 x^2)| \leqslant \tfrac{4}{3}\pi^3\, |x|^3 \tag{41}$$

for real x. We wish to apply this to the innermost exponential in (38). For concise notation, let us denote the argument of the exponential by the symbol $Z^{(\varepsilon)}(\xi, \tau)$:

$$Z^{(\varepsilon)}(\xi, \tau) \equiv \varepsilon^{-1/2} \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \phi(X^{(m)}(s) - \xi, (s-\tau)/\varepsilon). \tag{42}$$

$Z^{(\varepsilon)}$ depends on variables other than ξ and τ, but we do not explicitly indicate this dependence since the other variables may be considered fixed in what follows. Using now the Taylor expansion (41), we have

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-1} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi\, (e^{2\pi i Z^{(\varepsilon)}(\xi,\tau)} - 1)$$

$$= \lim_{\varepsilon \searrow 0} \varepsilon^{-1} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi\, 2\pi i Z^{(\varepsilon)}(\xi, \tau) - 2\pi^2 (Z^{(\varepsilon)}(\xi, \tau))^2 + O(|Z^{(\varepsilon)}(\xi, \tau)|^3), \tag{43}$$

with rigorous connotation of the order symbol O. Note that since ϕ is integrable, we should expect only an $O(\varepsilon)$ region of the s integral of $Z^{(\varepsilon)}$ in (42) to contribute significantly, and thus $Z^{(\varepsilon)}$ is formally an $O(\varepsilon^{1/2})$ quantity. This explains why we could expect to stop the Taylor expansion (43) after two terms.

We have three tasks before us:

1. First we shall show that the contribution from the error term is indeed negligible.

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-1} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \, |Z^{(\varepsilon)}(\xi, \tau)|^3 = 0. \tag{44}$$

2. Then we will show that the term linear in $Z^{(\varepsilon)}(\xi, \tau)$ vanishes exactly.

$$\varepsilon^{-1} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \, Z^{(\varepsilon)}(\xi, \tau) = 0. \tag{45}$$

3. Third, we evaluate the $\varepsilon \searrow 0$ limit of the quadratic term:

$$\lim_{\varepsilon \searrow 0} -2\pi^2 \varepsilon^{-1} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi (Z^{(\varepsilon)}(\xi, \tau))^2 = \int_{t'}^{t} U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{X^{(j)}(s)\}) \, ds. \tag{46}$$

where $U_{N,\mathrm{DR}}$ is defined in (40).

Upon completing these, we will have shown that

$$\lim_{\varepsilon \searrow 0} \hat{p}^{(\varepsilon)}_{N,\mathrm{D}}(\{\eta^{(j)}\}, \{k^{(j)}\}, t' \mid \{x^{(j)}\}, \{y^{(j)}\}, t)$$

$$= \hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}\}, \{k^{(j)}\}, t' \mid \{x^{(j)}\}, \{y^{(j)}\}, t)$$

where $\hat{\bar{p}}_{N,\mathrm{D}}$ is defined in (40). This is what we desired for this subsection. We are thus left to verify the three points outlined above.

7.1.1. *First Substep: Asymptotic Negligibility of Error Term*

To prove Eq. (44), change variables in the integral for $Z^{(\varepsilon)}(\xi, \tau)$ to $s' = s/\varepsilon$:

$$Z^{(\varepsilon)}(\xi, \tau) = \varepsilon^{1/2} \sum_{m=1}^{N} k^{(m)} \int_{t'/\varepsilon}^{t/\varepsilon} \phi(X^{(m)}(s') - \xi, s' - (\tau/\varepsilon)) \, ds'.$$

Clearly:

$$|Z^{(\varepsilon)}(\xi,\tau)| \leqslant \varepsilon^{1/2} \sum_{m=1}^{N} |k^{(m)}| \int_{-\infty}^{\infty} ds' \sup_{x\in\mathbb{R}} |\phi(x,s')| \leqslant C\varepsilon^{1/2} \tag{47}$$

for some constant C independent of ε. Using this inequality for two factors of $Z^{(\varepsilon)}(\xi,\tau)$ in the error term, we can bound:

$$\begin{aligned}
&\varepsilon^{-1}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi\, |Z^{(\varepsilon)}(\xi,\tau)|^3 \\
&\leqslant C^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi\, |Z^{(\varepsilon)}(\xi,\tau)| \\
&\leqslant C^2\varepsilon^{1/2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \int_{t'/\varepsilon}^{t/\varepsilon} \left| \sum_{m=1}^{N} k^{(m)}\phi(X^{(m)}(s')-\xi, s'-(\tau/\varepsilon)) \right| ds'.
\end{aligned} \tag{48}$$

Using Fubini's theorem, we integrate first over τ (which produces another factor of ε), then integrate over ξ (which produces a ε-independent constant), and then finally over s' (which just gives $(t-t')/\varepsilon$). In the end, we can bound (48) by $C'(t-t')\,\varepsilon^{1/2}$, where C is a constant independent of ε. We have thus shown that the error term in (43) is asymptotically negligible.

7.1.2. Second Substep: Vanishing of Linear Term

Next, the linear term in (43) vanishes as an immediate consequence of the fact that the integral of $\phi(x,t)$ over $\mathbb{R}\times\mathbb{R}$ vanishes.

7.1.3. Third Substep: Evaluation of Limit of Quadratic Term

We begin by writing down the meaning of $Z^{(\varepsilon)}$ (42):

$$\begin{aligned}
&-2\pi^2\varepsilon^{-1}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi (Z^{(\varepsilon)}(\xi,\tau))^2 \\
&\quad = -2\pi^2\varepsilon^{-2} \sum_{m=1}^{N}\sum_{n=1}^{N} k^{(m)}k^{(n)} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \int_{t'}^{t} ds \int_{t'}^{t} ds' \\
&\qquad \times \phi(X^{(m)}(s)-\xi, (s-\tau)/\varepsilon)\, \phi(X^{(n)}(s')-\xi, (s'-\tau)/\varepsilon).
\end{aligned} \tag{49}$$

Changing variables $\tau' = \tau/\varepsilon$ and using Fubini's theorem, we obtain

$$
\begin{aligned}
&-2\pi^2\varepsilon^{-1} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi (Z^{(\varepsilon)}(\xi,\tau))^2 \\
&\quad = -2\pi^2\varepsilon^{-1} \sum_{m=1}^{N} \sum_{n=1}^{N} k^{(m)} k^{(n)} \int_{t'}^{t} ds \int_{t'}^{t} ds' \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\xi \\
&\qquad \times \phi(X^{(m)}(s)-\xi, s/\varepsilon-\tau')\, \phi(X^{(n)}(s')-\xi, s'/\varepsilon-\tau') \\
&\quad = -2\pi^2\varepsilon^{-1} \sum_{m=1}^{N} \sum_{n=1}^{N} k^{(m)} k^{(n)} \int_{t'}^{t} ds \int_{t'}^{t} ds' \\
&\qquad \times \tilde{R}(X^{(m)}(s)-X^{(n)}(s'), (s-s')/\varepsilon).
\end{aligned} \tag{50}
$$

Now we take the $\varepsilon \searrow 0$ limit of (50). As $|\tilde{R}(x,t)| \leqslant \tilde{R}(0,0) < \infty$, we can commute the limit inside the first integral, and the resulting limit can be evaluated using Lemma 14 in Appendix B, since $\tilde{R}(x,t)$ is continuous with compact support and $X^{(j)}(s)$ is continuous in every realization. In this way, we obtain Eq. (46) and achieve our first goal of showing that the diffusively rescaled N-particle characteristic function $\hat{p}^{(\varepsilon)}_{N,\mathrm{D}}$ converges pointwise to $\hat{\bar{p}}_{N,\mathrm{D}}$ (Claim 5). We will show how this implies weak convergence of $p^{(\varepsilon)}_{N,\mathrm{D}}$ to $\bar{p}_{N,\mathrm{D}}$ and uniform convergence $P^{(\varepsilon)}_{N,\mathrm{D}}$ to $\bar{P}_{N,\mathrm{D}}$ on compact sets in Section 7.3, but first we show that $\hat{\bar{p}}_{N,\mathrm{D}}$, $\bar{p}_{N,\mathrm{D}}$, and $\bar{P}_{N,\mathrm{D}}$ solve the PDE's stated in Proposition 2.

7.2. Derivation of PDEs for DRDT Limit

The PDE (22) for $\hat{\bar{p}}_{N,\mathrm{D}}$ is obtained from (40) via the Feynman–Kac formula. Indeed,

$$
\begin{aligned}
&\hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) \\
&\quad \equiv e^{2\pi i (\sum_{j=1}^{N} k^{(j)} y^{(j)})} \Big\langle e^{2\pi i \sum_{j=1}^{N} \eta^{(j)} X^{(j)}(t')} \\
&\qquad \times \exp\left[\int_{t'}^{t} \left(-4\pi^2 \kappa \sum_{j=1}^{N} \left(k^{(j)}\right)^2 + U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{X^{(j)}(s)\}) \right) ds \right] \Big\rangle_W .
\end{aligned} \tag{51}
$$

where:

$$X^{(j)}(s) = x^{(j)} + \sqrt{2\kappa}\, W^{(j)}(t-s).$$

As $U_{N,\mathrm{DR}}$ is bounded and uniformly Hölder continuous as a function of $\{x^{(j)}\}$ (since $R(x)$ is), the Feynman–Kac formula (Appendix B) may be applied to deduce that $\hat{\bar{p}}_{N,\mathrm{D}}$ is the unique classical solution to the PDE:

$$\frac{\partial \hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)}{\partial t}$$
$$= \kappa \sum_{j=1}^{N} \frac{\partial^2 \hat{\bar{p}}_{N,\mathrm{D}}}{\partial (x^{(j)})^2} + \left(-4\pi^2\kappa \sum_{j=1}^{N} (k^{(j)})^2 + U_{N,\mathrm{DR}}(\{k^{(j)}\}, \{x^{(j)}\}) \right) \hat{\bar{p}}_{N,\mathrm{D}},$$
$$\hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t = t') = e^{2\pi i (\sum_{j=1}^{N} \eta^{(j)} x^{(j)} + k^{(j)} y^{(j)})}.$$

It is easily checked either from this PDE or from (51) that $\hat{\bar{p}}_{N,\mathrm{D}}$ is always equal to $e^{2\pi i(\sum_{j=1}^{N} k^{(j)} y^{(j)})}$ multiplied by a function independent of $y^{(j)}$. Hence, multiplication of $\hat{\bar{p}}_{N,\mathrm{D}}$ by $2\pi i k^{(j)}$ is precisely equivalent to differentiation $\frac{\partial}{\partial y^{(j)}}$. From this observation, we obtain the imaginary-time Schrödinger PDE for $\hat{\bar{p}}_{N,\mathrm{D}}$ reported in (22), as well as the fact that $\hat{\bar{p}}_{N,\mathrm{D}}$ satisfies the PDE (24a). To show that $\bar{p}_{N,\mathrm{D}}$ satisifes the PDE (24c), we express it as an inverse Fourier integral of the manifestly integrable function $\hat{\bar{p}}_{N,\mathrm{D}}$:

$$\bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)$$
$$= \int_{\mathbb{R}^{2N}} \hat{\bar{p}}_{N,\mathrm{D}}(\{\eta^{(j)}, k^{(j)}\}; t' \mid \{x^{(j)}, y^{(j)}\}; t)$$
$$\times e^{-2\pi i (\sum_{j=1}^{N} \eta^{(j)} x'^{(j)} + k^{(j)} y'^{(j)})} d\{\eta^{(j)}\}\, d\{k^{(j)}\}.$$

All that is required now is to justify that partial derivatives with respect to $\{x^{(j)}\}$, $\{y^{(j)}\}$ and t may be commuted inside the Fourier integral for $t > t'$. To do this, one can use Itô's formula to show that such derivatives of $\hat{\bar{p}}_{N,\mathrm{D}}$ (51) are bounded and integrable as a function of $\{\eta^{(j)}\}$ and $\{k^{(j)}\}$ and then apply a standard dominated convergence argument to finite difference approximations.

7.3. Convergence of Correlation Functions in DRDT Limit

It remains to prove the weak convergence of the N-particle transition densities $p^{(\varepsilon)}_{N,\mathrm{D}}$ to $\bar{p}_{N,\mathrm{D}}$ and the convergence of the $P^{(\varepsilon)}_{N,\mathrm{D}}(\{x^{(j)}, y^{(j)}\}, t)$ to the limiting form:

$$\bar{P}_{N,\mathrm{D}}(\{x^{(j)}, y^{(j)}\}, t) \equiv \int_{\mathbb{R}^{2N}} \bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t)$$
$$\times P_{N,0}(\{x'^{(j)}, y'^{(j)}\})\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N}.$$

But from (19), we can write:

$$P_{N,\mathrm{D}}^{(\varepsilon)}(\{x^{(j)}, y^{(j)}\}, t) = \int_{\mathbb{R}^{2N}} p_{N,\mathrm{D}}^{(\varepsilon)}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t) \times P_{N,0}(\{x'^{(j)}, y'^{(j)}\})\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N},$$

where $p_{N,\mathrm{D}}^{(\varepsilon)}$ is the N-particle transition density under diffusive rescaling. By definition, $p_{N,\mathrm{D}}^{(\varepsilon)}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}\, d\{y'^{(j)}\}$ is precisely the probability measure for the random variables:

$$X^{(j)}(0) \equiv x^{(j)} + \sqrt{2\kappa}\, W^{(j)}(t),$$

$$Y^{(j)}(0) \equiv y^{(j)} - \int_0^t v_{\mathrm{D}}^{(\varepsilon)}(X^{(j)}(s), s)\, ds + \sqrt{2\kappa}\, W_y^{(j)}(t),$$

and $\hat{p}_{N,\mathrm{D}}^{(\varepsilon)}(\{\eta^{(j)}, k^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t)$ is the characteristic function for these random variables. Here $\{W^{(j)}(t), W_y^{(j)}(t)\}_{j=1}^{N}$ is a collection of independent Wiener processes.

Levy's Continuity theorem (ref. 59, p. 130) states quite generally that the pointwise convergence of a family of characteristic functions, which is uniform in a neighborhood of the origin, implies weak convergence of the corresponding probability measures to some limiting probability measure. Moreover, this limiting probability measure has for its characteristic function the pointwise limit of the converging family of characteristic functions. Now, we have shown in Section 7.1 the pointwise convergence of the N-particle characteristic functions $\hat{p}_{N,\mathrm{D}}^{(\varepsilon)}(\{\eta^{(j)}, k^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t)$ to a limit $\hat{\bar{p}}_{N,\mathrm{D}}$. Because of Ascoli's Theorem (ref. 61, p. 169) and the boundedness and equicontinuity of $\hat{p}_{N,\mathrm{D}}^{(\varepsilon)}$ with respect to $\{\eta^{(j)}, k^{(j)}\}$, this convergence is, for each $\{x^{(j)}, y^{(j)}\}$ and t, uniform on compact sets of $\eta^{(j)}$ and $k^{(j)}$. Thus Levy's Continuity Theorem is in force, and we have that the probability measures $p_{N,\mathrm{D}}^{(\varepsilon)}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}\, d\{y'^{(j)}\}$ converge weakly to a probability measure $\bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t)\, d\{x'^{(j)}\}\, d\{y'^{(j)}\}$, which has characteristic function $\hat{\bar{p}}_{N,\mathrm{D}}$. Because of the boundedness and continuity of $P_{N,0}$, this weak convergence of measures implies that:

$$\begin{aligned}\lim_{\varepsilon \searrow 0} P_{N,\mathrm{D}}^{(\varepsilon)}(\{x^{(j)}, y^{(j)}\}, t) &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^{2N}} p_{N,\mathrm{D}}^{(\varepsilon)}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t) \\ &\quad \times P_{N,0}(\{x'^{(j)}, y'^{(j)}\})\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N} \\ &= \int_{\mathbb{R}^{2N}} \bar{p}_{N,\mathrm{D}}(\{x'^{(j)}, y'^{(j)}\}; 0 \mid \{x^{(j)}, y^{(j)}\}; t) \\ &\quad \times P_{N,0}(\{x'^{(j)}, y'^{(j)}\})\, d\{x'^{(j)}\}_{j=1}^{N}\, d\{y'^{(j)}\}_{j=1}^{N} \\ &\equiv \bar{P}_{N,\mathrm{D}}(\{x^{(j)}, y^{(j)}\}, t).\end{aligned}$$

To show that this convergence of $P^{(\varepsilon)}_{N,\mathrm{D}}$ to $\bar{P}_{N,\mathrm{D}}$ is uniform on compact sets, we use standard relations between smoothness and decay of Fourier transform pairs to extract uniform integrability and moduli of continuity estimates on $\bar{p}_{N,\mathrm{D}}$ for $t > t'$ from the explicit formula for $\hat{\bar{p}}_{N,\mathrm{D}}$ (51). These can be used to establish uniform convergence of the continuous functions $P^{(\varepsilon)}_{N,\mathrm{D}}$ to $\bar{P}_{N,\mathrm{D}}$ on compact sets away from $t = t'$. To achieve a demonstration of continuity and uniform convergence on compact subsets of $t \geqslant t'$ which touch $t = t'$, one writes further uniform moduli of continuity estimates using the continuity and boundedness of $P_{N,0}$ and the very rapid decay of $\bar{p}_{N,\mathrm{D}}$ as $t \searrow t'$ (which follows from the derivatives of $\hat{\bar{p}}_{N,\mathrm{D}}$ with respect to $\{\eta^{(j)}, k^{(j)}\}$ becoming very small). We omit the standard details.

8. DERIVATION OF PASSIVE SCALAR STATISTICS IN FIRDT LIMIT

In this section, we prove Proposition 3, in which we derived the passive scalar statistics in the FIRDT (fixed intensity rapid decorrelation in time limit). As in our proof of the DRDT limit, our main focus is on proving the pointwise convergence of the N-particle characteristic function $\hat{p}^{(\varepsilon)}_{N,\mathrm{FI}}$ associated to the velocity field rescaled with fixed intensity to a limit $\hat{\bar{p}}_{N,\mathrm{FI}}$ which obeys the PDE (28) stated in the proposition. At the end we mention how to infer from this the convergence of the N-particle transition densities $p^{(\varepsilon)}_{N,\mathrm{FI}}$ and the PS correlation functions $P^{(\varepsilon)}_{N,\mathrm{FI}}$ corresponding to fixed intensity rescaling.

8.1. Convergence of *N*-Particle Characteristic Function in FIRDT Limit

Under the fixed intensity rescaling of the Poisson blob shear velocity field, the blob placement intensity is held fixed at unity, and the blob function is rescaled according to (14). Thus, the N-particle characteristic function associated to the velocity field rescaled with fixed intensity is, according to Proposition 4 in Section 6, given for $t' < t$ by:

$$\hat{p}^{(\varepsilon)}_{N,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$$

$$= e^{2\pi i(\sum_{j=1}^{N} k^{(j)} y^{(j)}) - 4\pi^2 \kappa \sum_{j=1}^{N} (k^{(j)})^2 (t-t')} \Big\langle e^{2\pi i \sum_{j=1}^{N} \eta^{(j)} X^{(j)}(t')}$$

$$\times \exp\left[\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi \left(e^{-2\pi i \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \varepsilon^{-1} \phi(X^{(m)}(s)-\xi,\, (s-\tau)/\varepsilon)} - 1\right)\right]\Big\rangle_W . \tag{52}$$

with:

$$X^{(j)}(t') = x^{(j)} + \sqrt{2\kappa}\, W^{(j)}(t-t'). \tag{53}$$

The reader may wish to contrast the appearance of ε in this formula with the appearance of ε in the diffusively rescaled N-particle characteristic functions (38).

Our aim in this subsection is to prove:

Claim 6. For $t' < t$,

$$\lim_{\varepsilon \searrow 0} \hat{p}^{(\varepsilon)}_{N,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t) = \hat{\bar{p}}_{N,\mathrm{FI}}(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)$$

pointwise, where:

$$\begin{aligned}\hat{\bar{p}}_{N,\mathrm{FI}}&(\{\eta^{(j)}, k^{(j)}\}, t' \mid \{x^{(j)}, y^{(j)}\}, t)\\ &\equiv e^{2\pi i(\sum_{j=1}^{N} k^{(j)} y^{(j)}) - 4\pi^2 \kappa \sum_{j=1}^{N} (k^{(j)})^2 (t-t')}\\ &\quad\times \left\langle e^{2\pi i \sum_{j=1}^{N} \eta^{(j)} X^{(j)}(t')} \exp\left[\int_{t'}^{t} U_{N,\mathrm{FI}}(\{k^{(j)}\}, \{X^{(j)}(s)\})\, ds\right]\right\rangle_W.\end{aligned}$$

with:

$$U_{N,\mathrm{FI}}(\{k^{(j)}\}, \{x^{(j)}\}) = \int_{-\infty}^{\infty} (e^{-2\pi i \sum_{l=1}^{N} k_l \bar{\phi}(x_l - \xi)} - 1)\, d\xi,$$

$$\bar{\phi}(x) = \int_{-\infty}^{\infty} dt\, \phi(x, t)$$

To begin, the $\varepsilon \searrow 0$ limit may be taken inside the expectation of the Wiener process in (52) by the bounded convergence theorem. We want to bring the limit furthermore inside the integral over τ and ξ. To do this, we will use a generalized form of the Lebesgue's Dominated Convergence Theorem:

Lemma 7 (Generalized Lebesgue Dominated Convergence Theorem). Let a sequence of functions f_n approach a function f pointwise almost everywhere in the limit $n \to \infty$. Suppose that we can find a sequence of integrable functions g_n such that $|f_n| \leqslant g_n$, $\lim_{n\to\infty} g_n = g$, and $\lim_{n\to\infty} \int g_n = \int g$. Then $\lim_{n\to\infty} \int f_n = \int f$.

The proof may be found in ref. 61, p. 92.

We wish to find such a dominating sequence $g_\varepsilon(\xi, \tau)$ for the function:

$$e^{2\pi i \varepsilon^{-1} \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \phi(X^{(m)}(s)-\xi,\, (s-\tau)/\varepsilon)} - 1.$$

First, we use the simple geometric fact that the length of an arc of a circle is longer than its chord, which, when applied to the complex plane, tells us that:

$$|e^{2\pi i z} - 1| \leqslant 2\pi\, |z|$$

for real z. This gives us the estimate:

$$|e^{2\pi i \varepsilon^{-1} \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \phi(X^{(m)}(s)-\xi,\, (s-\tau)/\varepsilon)} - 1|$$

$$\leqslant 2\pi\varepsilon^{-1} \sum_{m=1}^{N} |k^{(m)}| \int_{t'}^{t} ds\, |\phi(X^{(m)}(s)-\xi,\, (s-\tau)/\varepsilon)| \equiv g_\varepsilon(\xi, \tau).$$

We can use Lemma 14 in Appendix B to evaluate the $\varepsilon \searrow 0$ limit of this function, since ϕ is continuous with compact support and $X^{(j)}(s)$ is continuous in every realization. Thus, we have:

$$g(\xi, \tau) \equiv \lim_{\varepsilon \searrow 0} g_\varepsilon(\xi, \tau)$$

$$= \begin{cases} 0 & \text{if} \quad \tau < s \quad \text{or} \quad \tau > t, \\ 2\pi \sum_{m=1}^{N} |k^{(m)}|\, \psi(X^{(m)}(\tau)-\xi) & \text{if} \quad s < \tau < t, \end{cases} \tag{54}$$

where

$$\psi(x) \equiv \int_{-\infty}^{\infty} |\phi(x, t)|\, dt.$$

The g_ε are thus a good dominating sequence if we can show:

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi\, \varepsilon^{-1} \int_{t'}^{t} ds\, |\phi(X^{(m)}(s)-\xi,\, (s-\tau)/\varepsilon)|$$

$$= \int_{t'}^{t} d\tau \int_{-\infty}^{\infty} d\xi\, \psi(X^{(m)}(\tau)-\xi).$$

But this is an easy consequence of Fubini's theorem:

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\xi\, \varepsilon^{-1} \int_{t'}^{t} ds\, |\phi(X^{(m)}(s)-\xi, (s-\tau)/\varepsilon)| \\
&\quad = \lim_{\varepsilon \searrow 0} \varepsilon^{-1} \int_{t'}^{t} ds \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\tau\, |\phi(X^{(m)}(s)-\xi, (s-\tau)/\varepsilon)| \\
&\quad = \lim_{\varepsilon \searrow 0} \int_{t'}^{t} ds \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\tau'\, |\phi(X^{(m)}(s)-\xi, s/\varepsilon-\tau')| \\
&\quad = \lim_{\varepsilon \searrow 0} \int_{t'}^{t} ds \int_{-\infty}^{\infty} d\xi\, \psi(X^{(m)}(s)-\xi).
\end{aligned}$$

We have therefore justified the passage of the $\varepsilon \searrow 0$ limit all the way inside the integral of the exponential. Our key task is thus to evaluate:

$$\lim_{\varepsilon \searrow 0} \varepsilon^{-1} \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \phi(X^{(m)}(s)-\xi, (s-\tau)/\varepsilon). \tag{55}$$

$\phi(x, t)$ is assumed continuous and uniformly integrable as a function of t, and $X^{(m)}(s)$ is continuous in every realization. Therefore, we may apply Lemma 14 in Appendix B:

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \varepsilon^{-1} \sum_{m=1}^{N} k^{(m)} \int_{t'}^{t} ds\, \phi(X^{(m)}(s)-\xi, (s-\tau)/\varepsilon) \\
&\qquad = \begin{cases} \bar{\phi}(X^{(m)}(\tau)-\xi) & \text{if} \quad t' < \tau < t, \\ 0 & \text{else,} \end{cases}
\end{aligned} \tag{56}$$

where:

$$\bar{\phi}(x) = \int_{-\infty}^{\infty} dt\, \phi(x, t). \tag{57}$$

Using this result, we obtain pointwise convergence of $\hat{p}^{(\varepsilon)}_{N,\mathrm{FI}}$ to $\hat{\bar{p}}_{N,\mathrm{FI}}$, as we desired to show in this subsection (Claim 6).

8.2. Derivation of PDE for FIRDT Limit

The fact that the limiting N-particle characteristic function $\hat{p}^{(\varepsilon)}_{N,\mathrm{FI}}$ is the classical solution of the PDE (28) follows directly from the Feynman–Kac formula in the same manner as we argued in Section 7.2 for the DRDT limit. One simply checks first that the potential $U_{N,\,\mathrm{FI}}(\{k^{(j)}\}, \{x^{(j)}\})$ is bounded and uniformly Hölder continuous on compact sets.

8.3. Convergence of Correlation Functions in DRDT Limit

The procedure to show that $p^{(\varepsilon)}_{N,\,\mathrm{FI}}$ converge weakly to $\bar{p}_{N,\,\mathrm{FI}}$ and $P^{(\varepsilon)}_{N,\,\mathrm{FI}}$ converges uniformly on compact sets to the continuous function $\bar{P}_{N,\,\mathrm{FI}}$ follows the same outline as in the DRDT Limit (Section 7.3).

APPENDIX A. PROPERTIES OF POISSON POINT PROCESS

A.1. Definition

Definition 8. The Poisson distribution for a random variable X with range consisting of non-negative integer values is defined:

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \tag{58}$$

where λ is a parameter equal to the mean of X.

Definition 9. A *Poisson point process* of *intensity* λ is defined as a random array of points in the space considered with the following properties:

- The number of points situated in two nonoverlapping regions A and B are independent of each other.
- The number of points distributed in a Borel set A is given by a random variable obeying the Poisson distribution with mean $\lambda m(A)$, where $m(A)$ is the Lebesgue measure of the set A.

A.2. Functional Averages Involving Poisson Point Process

The *characteristic functional* (ref. 62, p. 282) of a Poisson point process is given by:

$$\phi(F) = \langle e^{2\pi i \sum_n F(x_n)} \rangle \tag{59}$$

where the x_n are described by a Poisson point process of intensity λ. The characteristic functional is the random process or random field analogue of the characteristic function of a random variable.

The characteristic functional for a Poisson point process can be evaluated in closed form:

Proposition 10 (Characteristic Functional, Poisson). If x_n is a Poisson point process of intensity λ on a domain D, then its characteristic function is given by:

$$\langle e^{2\pi i \sum_n F(x_n)} \rangle = \exp\left[\lambda \int_D dx (e^{2\pi i F(x)} - 1) \right], \tag{60}$$

where F is an arbitrary continuous, bounded function which is integrable on D.

Two consequences of this formula which will also be useful are:

Corollary 11. If x_n is a Poisson point process of intensity λ on a domain D, then for any continuous, bounded, integrable function F on D,

$$\left\langle \sum_n F(x_n) \right\rangle = \lambda \int_D dx\, F(x) \tag{61}$$

Corollary 12. If x_n is a Poisson point process of intensity λ on a domain D, then for any continuous, bounded, integrable functions F and G defined on D,

$$\left\langle \sum_n F(x_n) \sum_{n'} G(x_{n'}) \right\rangle$$

$$= \lambda \int_D dx\, F(x)\, G(x) + \lambda^2 \left(\int_D dx\, F(x) \right) \left(\int_D dx'\, G(x') \right). \tag{62}$$

A.2.1. Proof of Proposition

To prove (60) rigorously, one can first verify this formula for the case where F is a smooth function of compact support; this is done in ref. 62, p. 282. As such functions are dense in $C(D) \cap L^1(D)$, the proof can be completed via the dominated convergence theorem.

A.2.2. Heuristic Derivation of Proposition

To provide some insight into the formula, we shall also outline a heuristic derivation, skipping technical details. First, the characteristic function for a Poisson random variable X with mean λ is found by a trite calculation:

$$\langle e^{2\pi i\xi X}\rangle = \sum_{k=0}^{\infty} e^{2\pi ik\xi}P(X=k) = \sum_{k=0}^{\infty} e^{2\pi ik\xi}e^{-\lambda}\frac{\lambda^k}{k!} = \exp[\lambda(e^{2\pi i\xi}-1)]. \qquad (63)$$

Now imagine that we partition D into little regions D_k over which $F(x)$ does not vary much. The number of Poisson points within each interval give a sequence of independent random variables, and thus we may factorize the expectation appearing in (59) as a product of expectations over each region D_k. Let F_k denote an average value of $F(x)$ over D_k; then:

$$\phi(F) \approx \prod_k \langle e^{2\pi iF_k |x_n \in D_k|}\rangle \qquad (64)$$

But the random variable counting the number of Poisson points in the domain D_k, $|x_n \in I_k|$, is given by a Poisson distribution with mean $\lambda m(D_k)$. Then as the number of the domains D_k increases and their sizes shrink, we have:

$$\phi(F) \approx \prod_k \exp[\lambda m(D_k)(e^{2\pi iF_k}-1)] \approx \exp\left[\int_D dx\, \lambda(e^{2\pi iF(x)}-1)\right]. \qquad (65)$$

A.2.3. Proof of Corollaries

For the first corollary (Corollary 11), introduce the real parameter μ; (60) tells us that:

$$\langle e^{2\pi i\mu \sum_n F(x_n)}\rangle = \exp\left[\lambda \int_D dx(e^{2\pi i\mu F(x)}-1)\right].$$

Evaluation of the derivative of each side with respect to μ at $\mu = 0$ gives the formula in the corollary. The possibility of commuting the derivative past the expectation on the left hand side and the integral on the right hand is guaranteed by the Lebesgue dominated convergence theorem, thanks to the boundedness and integrability of F.

Similarly, for the second corollary (Corollary 12), we introduce the real parameters μ and γ, and write from (60):

$$\langle e^{2\pi i \sum_n (\mu F(x_n)+\gamma G(x_n))}\rangle = \exp\left[\lambda \int_D dx(e^{2\pi i(\mu F(x)+\gamma G(x))}-1)\right].$$

Differentiation of each side with respect to μ and γ, and subsequent evaluation at $\mu = \gamma = 0$, will yield the result stated in the corollary. Note that $F, G \in L^2(D)$ since they are each absolutely integrable and bounded.

APPENDIX B. USEFUL LEMMAS

Here we record two mathematical statements which are used in the derivation of both the DRDT and FIRDT limits of the passive scalar statistics. The first is the Feynman–Kac formula, and the second is a general statement concerning the limiting behavior of the integral of a function which has a rapid decorrelation in time rescaling.

B.1. Feynman–Kac formula

Proposition 13. Under the conditions listed at the end of the theorem, the Cauchy problem:

$$\begin{aligned} \frac{\partial \psi(\mathbf{x}, t)}{\partial t} &= K \Delta_{\mathbf{x}} \psi(\mathbf{x}, t) + c(\mathbf{x}, t)\, \psi(\mathbf{x}, t), \\ \psi(\mathbf{x}, t=0) &= \psi_0(\mathbf{x}) \end{aligned} \tag{66}$$

on $\mathbb{R}^N \times \mathbb{R}_+$ has a unique classical solution given by the "path integral" formula:

$$\psi(\mathbf{x}, t) = \left\langle \psi_0(\mathbf{X}_{\mathbf{x},t}(0)) \exp\left(\int_0^t ds\, c(\mathbf{X}_{\mathbf{x},t}(s), s) \right) \right\rangle_W, \tag{67}$$

with:

$$\mathbf{X}_{\mathbf{x},t}(s) = \mathbf{x} + \sqrt{2K}\, \mathbf{W}(t-s), \tag{68}$$

where $\mathbf{W}(t)$ is a N-dimensional Wiener process. Sufficient smoothness conditions on the coefficients and initial data are:

- $K > 0$,
- $c(\mathbf{x}, t)$ is bounded and uniformly Hölder continuous on compact sets,
- $\psi_0(\mathbf{x})$ is bounded and continuous.

For a proof, see Theorems 6.4.6 and 6.5.2 of ref. 63.

B.2. Auxiliary Rapid Decorrelation Lemma

Lemma 14. Let $\psi(x,t)$ be a continuous function of compact support on $\mathbb{R}\times\mathbb{R}$. Define:

$$\bar{\psi}(x)=\int_{-\infty}^{\infty} dt\, \psi(x,t).$$

Then for any continuous function $g(t)$, and finite values of $t_1<t_2$,

$$\lim_{\varepsilon\searrow 0} \varepsilon^{-1}\int_{t_1}^{t_2} dt\, \psi(x+g(t),(t-s)/\varepsilon)=\begin{cases}\bar{\psi}(x+g(s)) & \text{for } t_1<s<t_2,\\ 0 & \text{for } s<t_1 \text{ or } s>t_2.\end{cases} \tag{69}$$

The idea of the lemma is that $\varepsilon^{-1}\psi(x,t/\varepsilon)$ should approach $\bar{\psi}(x)\,\delta(t)$ in the sense of generalized functions as $\varepsilon\searrow 0$. This lemma certainly holds for more general functions $\psi(x,t)$ which are uniformly integrable along constant x slices, but we only require consideration of functions of compact support for the purposes of this paper.

Proof of Lemma 14. Change integration variables to $u=\frac{t-s}{\varepsilon}$:

$$\varepsilon^{-1}\int_{t_1}^{t_2} dt\, \psi(x+g(t),(t-s)/\varepsilon)=\int_{\frac{t_1-s}{\varepsilon}}^{\frac{t_2-s}{\varepsilon}} du\, \psi(x+g(s+\varepsilon u),u). \tag{70}$$

Now if $s<t_1$, we have:

$$\left|\int_{t_1}^{t_2} dt\, \psi(x+g(t),(t-s)/\varepsilon)\right|\leqslant\int_{\frac{t_1-s}{\varepsilon}}^{\frac{t_2-s}{\varepsilon}} du \sup_{x\in\mathbb{R}}|\psi(x,u)|\to 0 \qquad \text{as} \quad \varepsilon\searrow 0$$

because ψ has compact support. A similar statement holds for $s>t_2$. Thus, we have proven the lemma for $s<t_1$ and for $s>t_2$.

We now turn to the substantial case $t_1<s<t_2$. Let $\gamma>0$ be a given small positive number. Because ψ has compact support, there exists $M<\infty$ so that:

$$|\psi(x,u)|=0 \qquad \text{for} \quad |u|\geqslant M. \tag{71}$$

The continuity of $\psi(x,u)$ and $g(t)$, along with the compact support of ψ, implies that there exists $\delta(\gamma)>0$ so that:

$$\sup_{|u|\leqslant M}\ \sup_{x\in\mathbb{R}} |\psi(x+g(u+u'),u)-\psi(x+g(u),u)|<\gamma/2M \qquad \text{for} \quad |u'|<\delta. \tag{72}$$

It follows that for:

$$0 < \varepsilon < \min\left(\frac{s-t_1}{M}, \frac{t_2-s}{M}, \frac{\delta(\gamma)}{M}\right),$$

we have:

$$\begin{aligned}
&\left|\int_{\frac{t_1-s}{\varepsilon}}^{\frac{t_2-s}{\varepsilon}} du\, \psi(x+g(s+\varepsilon u), u) - \int_{-\infty}^{\infty} du\, \psi(x+g(s), u)\right| \\
&\quad = \left|\int_{-M}^{M} du\, \psi(x+g(s+\varepsilon u), u) - \int_{-M}^{M} du\, \psi(x+g(s), u)\right| \\
&\quad \leqslant \int_{-M}^{M} du\, |\psi(x+g(s+\varepsilon u), u) - \psi(x+g(s), u)| \\
&\quad < 2M\, \frac{\gamma}{2M} = \gamma.
\end{aligned} \tag{73}$$

Since $\gamma > 0$ was arbitrary, we have:

$$\lim_{\varepsilon \searrow 0} \int_{\frac{t_1-s}{\varepsilon}}^{\frac{t_2-s}{\varepsilon}} du\, \psi(x+g(s+\varepsilon u), u) = \int_{-\infty}^{\infty} du\, \psi(x+g(s), u) = \bar{\psi}(x+g(s)) \tag{74}$$

by definition of $\bar{\psi}$. This concludes the proof of the lemma.

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